THE LONG-PERIOD EFFECTS OF PERTURBATION BY JUPITER ON THE ORBITS OF MINOR PLANETS WITH COMMENSURABLE MEAN MOTIONS

Fiona Vincent

A Thesis Submitted for the Degree of PhD
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University of St Andrews

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by

Fiona Vincent B.Sc.

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Preface

In October 1967, I matriculated at the University of St Andrews, and read for a joint degree in astronomy and pure mathematics, graduating in June 1971 with the degree of Bachelor of Science with honours of the second class, division I. In October 1971 I was admitted as a candidate for the degree of Doctor of Philosophy, under Resolution of the University Court 1967, no. 1. In August 1974 I took up employment outwith the University, and my research became part-time. Since October 1971 my research has been supervised by Mr T B Slebarski, lecturer in astronomy at the University Observatory.

Declaration

I declare that the following thesis is the result of work carried out by me, that the thesis is my own composition, and that it has not been presented in any previous application for a higher degree.

Fiona Vincent

Certificate

I certify that Fiona Vincent has satisfied the conditions of the Ordinance and Regulations, and that she is thus qualified to submit the accompanying thesis in application for the Degree of Doctor of Philosophy.

T.B.Slebarski

The heavens declare the glory of God,

and the firmament showeth His handiwork.

The Long-Period Effects of Perturbation by Jupiter on the Orbits of Minor Planets with Commensurable Mean Motions.

Abstract

Six minor planets with mean motions approximately commensurable with that of Jupiter were observed photographically: (87) Sylvia, (107) Camilla, (414) Liriope, (909) Ulla and (1574) Meyer (all with ratio 9/5), and (334) Chicago (ratio 3/2). The plates were measured on a two-coordinate measuring-machine, and reduced by the method of plate constants. Orbits were derived for all except (1574) Meyer; for (334) Chicago and for (909) Ulla, two orbits were obtained for successive years, though in the former case the orbits were not accurate.

The orbits of all six planets were also integrated numerically by computer, together with that of (153) Hilda, using a technique applied to the latter planet by J. Schubart. Based on the three-body problem, this applies three mutually-perpendicular components of the disturbing force to the orbital parameters, which consequently vary with time. Short-period variables are eliminated by averaging: only the longitudes of the two planets are allowed to vary, around one cycle of the commensurability. The derivatives of the orbital elements are calculated at intervals around the cycle, and their average values used in the integration.

Schubart's original method neglects the orbital inclinations of both planets; this investigation extends the equations to include the inclinations. His results for (153) Hilda are reproduced, and shown not to change greatly whether the inclinations are included or omitted. For (909) Ulla, however, considerable differences are found.

The results of the integrations on all seven orbits are presented in the form of graphs, covering time-intervals of 3000 years and more. The orbit of (909) Ulla was integrated twice, once taking the ratio of

mean motions as 9/5, and again as 7/4. The integrations of (153) Hilda and (334) Chicago took the ratio as 3/2. All the orbits considered appear stable over long time-intervals. The perihelion advance of (334) Chicago is interrupted by long retrogressions, as noted by Schubart. The orbit of (909) Ulla shows similar, though less marked, behaviour. Both planets are somewhat removed from exact commensurability.

Comparison was made between the orbits of four of the planets, derived from observations, and the orbits predicted by the numerical integrations. In most cases, the orbital parameters changed in the direction predicted but to a greater degree, indicating that more variables affect the real orbits of the minor planets than are taken into account in this theoretical investigation.

Fiona Vincent

October, 1979.

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I am also grateful to many other members of the Observatory, past and present, both for helpful discussions and for practical assistance. In particular I am obliged to Brian Yare for help with the early stages of the computer programming, and to Roger Stapleton for providing continued, unfailing help and support throughout, both with the computing and with all other matters.

I am also grateful to many members of the staff of St Andrews University Computing Laboratory for their assistance and for the computing-time afforded me there; to the computing centre at Cambridge University; and to the staff of Aberdeen University Computer Centre, where most of the calculations were performed and where this thesis was prepared.

Finally, and most importantly, my thanks go to my supervisor, Mr. T. B. Slebarski, for suggesting the topic of research, and for his assistance and kindly encouragement over the years.

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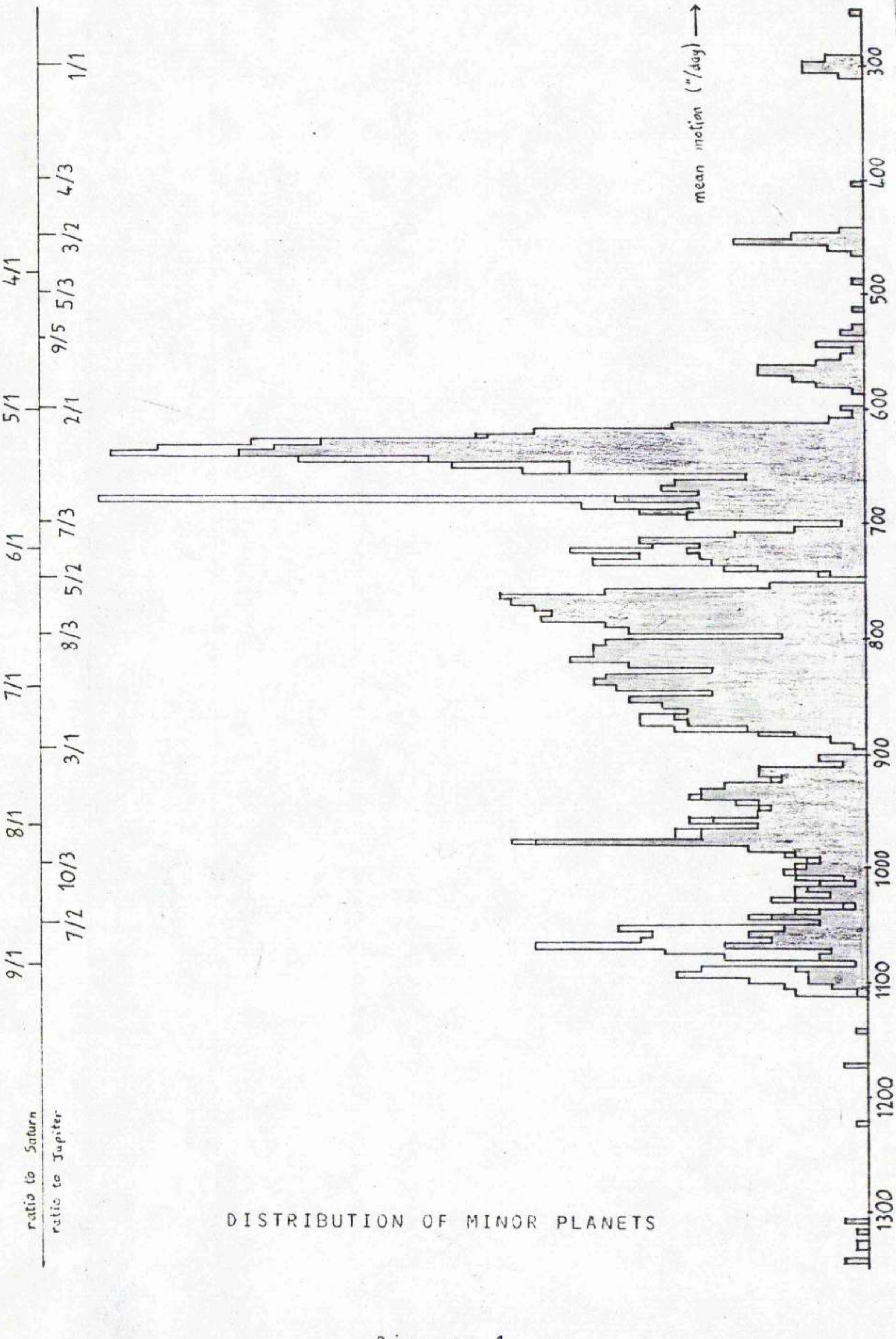
I - Introduction

The minor planets are mainly confined to that zone of the solar system lying between the orbits of Mars and Jupiter. They are not evenly distributed, as can be seen from diagram 1, which shows the distribution of the minor planets against their mean motions (which are directly related to the semi-major axes of their orbits). They are found at some heliocentric distances in great abundance, and hardly at all at others - these deficiencies are known as the Kirkwood Gaps.

The distribution is linked with the ratio of the mean motion of a minor planet at a particular heliocentric distance to the mean motion of Jupiter. Where this ratio is equivalent to the ratio of two integers P/Q, the mean motions are said to be commensurable. In particular, where P and Q are small, there is usually a significant feature in the distribution of the minor planets. In the case of the commensurabilities 3/1 and 5/2, for example, there is a sharply-defined gap, whereas peaks in the distribution occur at 3/2 and at 1/1. (The peak at 1/1 consists of the "Trojan asteroids", associated with the Lagrangian points on Jupiter's orbit, forming equilateral triangles with Jupiter and the sun.)



DISTRIBUTION OF MINOR PLANETS

Diagram 1

Diagram 1 shows some of the principal commensurabilities with both Jupiter and Saturn. (The unshaded portions of the diagram represent "families" of minor planets, as established by K. Hirayama and detailed by Brouwer (1951); these have probably resulted from the break-up of larger bodies, and have no relation to the mathematics of this problem.)

The nature of the effects of these commensurabilities has been the subject of much study (see, for example, Poincaré (1902), Hirayama and Akiyama (1937), Brouwer (1963), Schubart (1964, 1968, 1978), and Schweizer (1969)). Brouwer (1963) has demonstrated that a commensurability should always cause a V-shaped dip in an otherwise smooth distribution of asteroids; he conjectures that the commensurabilities are so closely clustered in the outer regions of the minor planet zone (i.e. as P/Q approaches 1) that the planets are forced into peaks in between them. Schweizer (1969) attempts to find a purely statistical explanation of the gaps. He suggests that a minor planet might cross a commensurability, showing as it does so a strong resonance in its orbital parameters with those of Jupiter; but he doubts if any such planet can remain at the commensurability for a long period.

It is not certain how much effect any particular commensurability has, nor how far its effect is felt by minor planets with mean motions differing slightly from exact commensurability; intuitively, one would expect that ratios of small numbers would have a greater effect than ratios of large numbers. It is also likely that commensurabilities with the mean motion of Saturn (shown in diagram 1) have some effect.

This investigation is mainly concerned with the orbits of minor planets in the vicinity of the 9/5 commensurability with Jupiter. These are on the outer fringe of the minor planet distribution; apart from the concentrations of planets previously mentioned, at 3/2 and at 1/1, there are only a handful of isolated planets further out. There are eight planets which have mean motions approximately 9/5 times that of Jupiter. Two more a little further out were not studied, but a third one was included in the investigation. This is (334) Chicago, which has been studied by several other authors, and which is generally regarded as an outlying member of the Hilda group, which have mean motions near 3/2 times that of Jupiter. This, together with five of the eight planets at the 9/5 commensurability, was observed photographically, and orbits were determined for all but one of them. These orbits are given in section II.

The theoretical investigation of the orbits of minor planets with mean motions commensurable with that of Jupiter presents certain well-established problems. For other minor planets, it is possible to develop the orbital parameters, which change under the gravitational perturbations of Jupiter (and, if desired, of any other major planets), in the form of infinite series; the independent variable may be time, or a time-dependent variable such as the planet's longitude. These series are assumed to be convergent, and so can be truncated after a sufficient number of terms to produce the accuracy required.

In the case of commensurable mean motions, the assumption of convergence cannot be maintained. This is because each term in such a series is of the general form

$$\frac{k_1}{j\ n + j'\ n_J} (\sin(j\ n + j'\ n\ )\ t + k_2\ )$$

where k_1 and k_2 are constants, where n and n_J are the mean motions of the minor planet and of Jupiter, and where j and j' are any integers. (The constant k_1 decreases rapidly with increasing values of j and j' .) If the mean motions are in the ratio P/Q, where P and Q are integers, then clearly setting j = Q and j' = -P will make the expression $(j\ n + j'\ n_J)$ very small. So this term, and any other term where j and j' are multiples of these, will be very large. This means that the series cannot be arbitrarily truncated at any point. (See, for example, Brouwer and Clemence (1961).)

This mathematical argument can be translated into physical terms, loosely, as follows. In the case of an "ordinary" minor planet, there is no simple ratio between its mean motion and that of Jupiter, and consequently no simple ratio between their periods. This means that, over a long period of time, every possible configuration of sun, Jupiter and minor planet is equally likely to occur; and the gravitational effect of Jupiter is equally likely to operate in any direction. So, over a long period of time, the perturbing effects "cancel out".

In the case of a minor planet at a commensurability, however, the periods of the planet and of Jupiter are in a ratio Q/P, and so a period of time covering P orbits of the minor planet also covers Q orbits of Jupiter. This means that any configuration of sun, Jupiter and minor planet that occurs will recur with this period. Thus the same limited set of configurations keeps recurring, and other configurations never occur at all; consequently the gravitational effects of Jupiter act always in only a limited set of ways, over any length of time.

It might seem, then, that the cumulative effects of Jupiter over a period of time on such a minor planet should eventually change its orbit, to the extent that it would no longer be at the commensurability; this may indeed be the explanation for the Kirkwood Gaps. However, this certainly does not apply in the case of the "Trojans", which, as pointed out above, form a cluster around the commensurability 1/1 . These exhibit only small perturbations in their orbits, causing them to "librate" around the Lagrangian points, and can be shown by classical mechanics to be stable (for example, Roy 1978). No simple theory exists, however, to analyse the stability of orbits at other commensurabilities, and numerical methods have to be used instead.

Since the normal method of expanding and truncating the varying orbital parameters cannot be applied, other methods have been sought and applied to various different commensurabilities. The method used here is that described by Schubart (1968), derived from the work of Poincaré (1902) and Schwarzschild (1903). Basically, it avoids the need for a series development, by a simple process of numerical integration, in which the orbit is established at one point in time, and the forces on the planet calculated; these give rise to changes in the orbit, which are assumed to be constant for a fixed time-interval. From these a new orbit is calculated for the next step of the integration. Normally, this type of integration is of limited usefulness in planetary orbits. A short time-step must be used, in order that the changes in the orbit can reasonably be expected to be constant throughout it; and the numerical errors, which accumulate in any numerical integration process, increase with the total number of steps taken, and consequently limit the number of steps which can be used. So the process, as described, can only be applied for a limited time-period.

The distinctive feature of the method used by Schubart (1968) is to limit the integration to variables which are only changing slowly. This means that a long time-step can be used in the integration, and consequently that it can be carried out over a much longer time-period. The way in which this is achieved is to separate the rapidly-changing variable, the longitude of the minor planet, and to average out the short-period effects over an interval equivalent to the period of the commensurability - i.e. over Q orbits of the minor planet, or P orbits of Jupiter. During this interval, the other (slowly-changing) orbital parameters are kept constant. The gravitational forces, and their effects on these slowly-changing parameters, are calculated at a set of equidistant intervals around this interval; the average of these effects is then applied to these variables, as one step in the main integration.

The theory used to calculate the gravitational forces is a version of the "restricted three-body problem" first defined by Poincaré. It is assumed that the two major bodies, Jupiter and the sun, move in undisturbed orbits around one another. They are not confined to circular orbits, as in the classical "restricted problem"; the eccentricity of Jupiter is included but is kept constant. The third body - the minor planet - is regarded as being of negligible mass, and consequently it moves under the gravitational attraction of the other two bodies without influencing their motion.

Schubart (1968) developed the equations for his averaging method in order to apply it to minor planets of the "Hilda group" - those around the commensurability 3/2. These planets are characterised by large orbital eccentricities and small inclinations. Schubart therefore neglected the inclination altogether and treated the problem as entirely two-dimensional. The present investigation was concerned with planets around the 9/5 commensurability, which have notably large inclinations to the ecliptic. For this reason, Schubart's method was extended into three dimensions, and the equations were redeveloped to take into account the inclination of the minor planet and also the inclination of Jupiter, which, like the eccentricity, was assumed to be constant. These equations proved to be highly complex, but reduced to those given by Schubart (1968) when the inclinations were set to zero.

The numerical integrations were carried out by computer. The programs for this were originally developed on the IBM 360/44 computer at the St Andrews University Computing Laboratory. As the actual integrations were extremely lengthy, the main integration program was transferred to the IBM 370/165 at Cambridge University, where it was run by means of a telephone link from St Andrews. The resultant data were transmitted back to St Andrews, also by telephone link, and were stored magnetically at the St Andrews Computing Laboratory for further processing. Unfortunately it frequently happened that some of the data were garbled in transmission; it also took several days to complete one integration. For these reasons, the programs were subsequently transferred to the Honeywell 66/80 machine at Aberdeen University Computer Centre, the data being stored and processed entirely in Aberdeen. The resultant listings and graphs were sent to St Andrews by 'bus; in general, one integration could be completed in one or two overnight runs, and all the results would be received a day or so later. Thus, most of the integrations discussed here were performed in Aberdeen.

During the development of the computer programs, the orbit of (153) Hilda was used, to permit a comparison with Schubart's results. It was found that good agreement was achieved, both with the inclinations set to zero, and also with the true values of the inclinations. Thus Schubart was entirely justified in neglecting the inclinations in his study. On a planet such as (909) Ulla, however, which has an inclination of some 18 degrees, the inclusion or omission of the inclinations did make a considerable difference to the results.

The results of the integrations were plotted out to permit an immediate understanding of the behaviour of the various orbital parameters over the long periods of time covered. These graphs appear in section III and are discussed in detail. In general, it appears that all the minor planets investigated have stable orbits; the parameters undergo fluctuations of various periods but of limited amplitudes, over the intervals of time considered (several thousands of years).

However, it is not certain how closely the behaviour of these numerically-integrated orbits parallels that of the real ones. For planets which could plausibly be assumed to be influenced by either of two commensurabilities, changing the parameters of the integration from one commensurability to the other causes considerable changes in the behaviour of the orbit. Other factors, such as the gravitational influence of Saturn, have also been neglected. These problems are discussed in section IV, where the numerically-integrated orbits are compared with those determined from observations.

Since the commencement of this investigation, Schubart has announced that he also has extended his original equations to take account of the inclination of the orbits (Schubart 1978); he has applied them to (153) Hilda, and also to various theoretical model orbits, on the basis of the circular restricted three-body problem. He does not give the actual equations; it appears that there are slight differences between his approach and that given here. His description of the results of his integration of the orbit of (153) Hilda, including the inclination, agrees qualitatively with the results found here.

II - Observations

The planets investigated were all in the vicinity of the 9/5 commensurability; i.e. $n/n_♃ = 1.80$. This corresponds to a semi-major axis of 3.52 A.U. The following planets were considered.

planet	a	$n/n_♃$
(87) Sylvia	3.481	1.826
(107) Camilla	3.490	1.820
(536) Merapi	3.504	1.809
(414) Liriope	3.505	1.809
(1328) Devota	3.507	1.806
(1574) Meyer	3.533	1.786
(909) Ulla	3.551	1.773
(721) Tabora	3.553	1.773
(522) Helga	3.628	1.716
(1144) Oda	3.752	1.632
(334) Chicago	3.885	1.549

The planets (522) Helga and (1144) Oda were rejected as being too far from the point of commensurability. However, the planet (334) Chicago was retained, as it has been investigated by several other authors; it is generally regarded as being on the fringe of that concentration of minor planets known as the Hilda group, which have $n/n_♃$ approximately equal to 3/2. Of the remaining planets, five were selected for observation: (87) Sylvia, (107) Camilla, (414) Liriope, (909) Ulla and (1574) Meyer. Their orbits were taken from the "Ephemerides of Minor Planets", and are given below.

(87) Sylvia

t = 0h 2 Dec 1962
$M = 211^\circ.064$
$\omega = 265^\circ.558$
$\Omega = 74^\circ.049$
$i = 10^\circ.853$
$\phi = 5^\circ.655$
$(e = 0.099)$
$a = 3.4812$

(107) Camilla

t = 0h 15 Jan 1947
$M = 291^\circ.867$
$\omega = 303^\circ.571$
$\Omega = 174^\circ.664$
$i = 9^\circ.922$
$\phi = 4^\circ.011$
$(e = 0.070)$
$a = 3.4895$

(334) Chicago

t = 0h 15 Jan 1947
$M = 59^\circ.014$
$\omega = 163^\circ.717$
$\Omega = 131^\circ.524$
$i = 4^\circ.626$
$\phi = 3^\circ.267$
$(e = 0.057)$
$a = 3.8854$

(414) Liriope

t = 0h 2 Dec 1962
$M = 119^\circ.213$
$\omega = 313^\circ.746$
$\Omega = 111^\circ.890$
$i = 9^\circ.542$
$\phi = 4^\circ.409$
$(e = 0.077)$
$a = 3.5072$

(909) Ulla

t = 0h 2 Dec 1962
$M = 316\overset{\circ}{.}241$
$\omega = 232\overset{\circ}{.}102$
$\Omega = 147\overset{\circ}{.}197$
$i = 18\overset{\circ}{.}795$
$\phi = 5\overset{\circ}{.}345$
$(e = 0.093)$
$a = 3.5513$

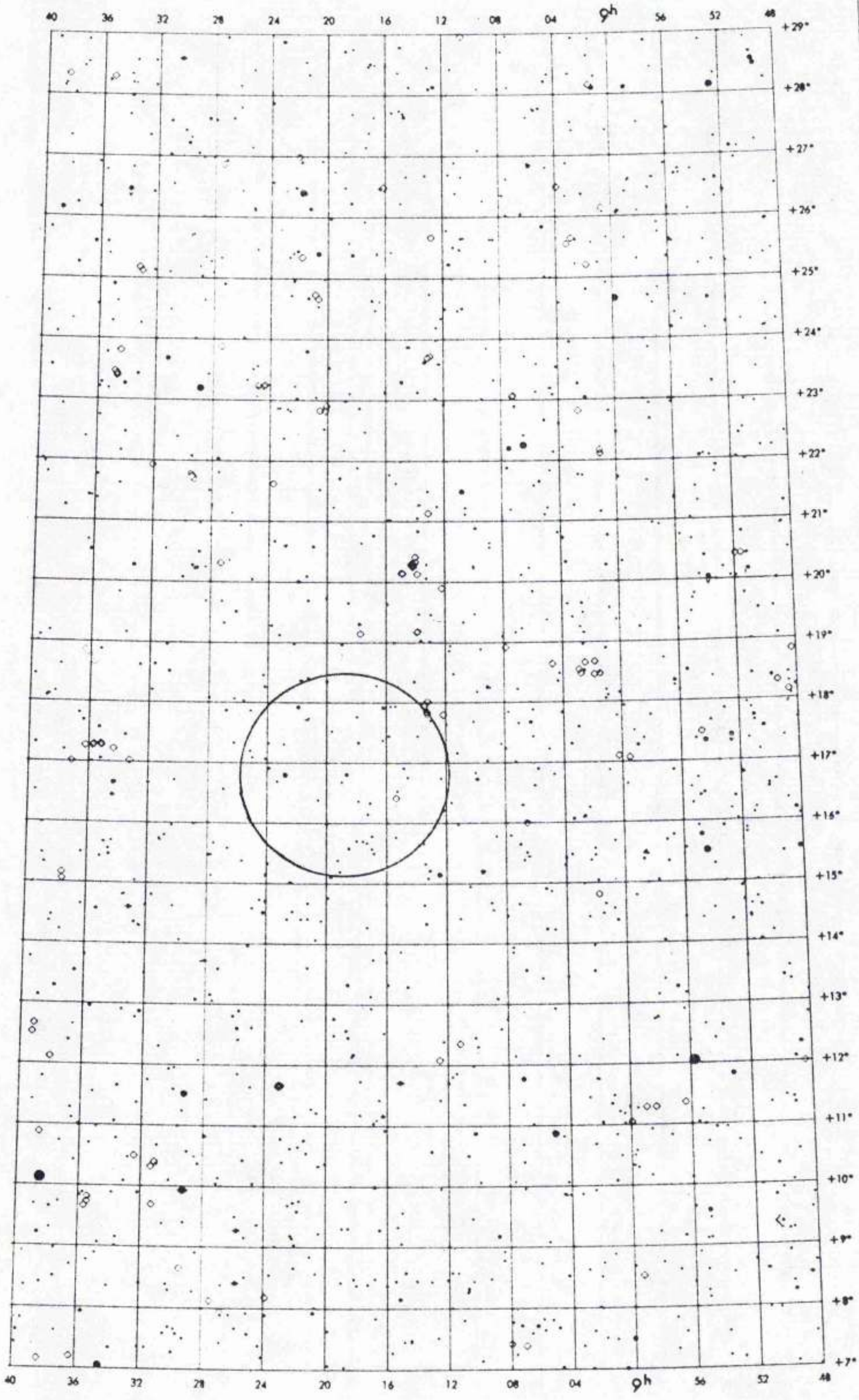
(1574) Meyer

t = 0h 15 Jan 1947
$M = 267\overset{\circ}{.}602$
$\omega = 269\overset{\circ}{.}954$
$\Omega = 247\overset{\circ}{.}291$
$i = 14\overset{\circ}{.}446$
$\phi = 2\overset{\circ}{.}692$
$(e = 0.047)$
$a = 3.5336$

This small group of planets forms a strong contrast with the planets of the Hilda group, which generally have low orbital inclinations but high eccentricities. Of the twenty or so members of that group, only two have inclinations of over 10 degrees; but only three of them have eccentricities under 0.1. The group of planets investigated here, on the other hand, have generally low eccentricities (ranging between 0.05 and 0.1) and high inclinations (ranging between 9.5 and 18.8 degrees). These differences have interesting implications for the theoretical study of the orbits.

Once the minor planets to be observed had been chosen, their opposition dates were obtained from the current "Ephemerides of Minor Planets". Each planet was observed for a period of about ten weeks around opposition, provided that this occurred during the observing season at St. Andrews, which is from mid-August to early May. (Astronomical twilight lasts all night for the remainder of the year.) Furthermore, observations were only made during the "dark" half of each month, from the last quarter of the moon to the first quarter. During the "bright" half of the month, it was likely that the moonlight would swamp the image of the minor planet on the photographic plate, since the planets used were all fairly faint - apparent magnitudes ranging between 12 and 16. (The telescope was used during the "bright" half-months by other researchers using photo-electric techniques.) Observations were made during three successive seasons, between August 1971 and May 1974.

Before each fortnight of observations, a sequence of preparations was made. The approximate positions of the minor planet(s) to be observed were obtained from the "Ephemerides of Minor Planets" (where they are listed at ten-day intervals) and plotted on to a star-chart; photo-copies of the appropriate pages of the Smithsonian Astrophysical Observatory (SAO) Star Atlas were used as charts.



SMITHSONIAN ASTROPHYSICAL OBSERVATORY

STAR-CHART (plate 43)

Diagram 2

(One such chart is reproduced in diagram 2, with a circle added to show the area covered by one plate - in this, case plate no. 43.) A sequence of guide-stars was then chosen, each lying as close as possible to the path of the minor planet. A circle of radius 0.75 degrees was drawn around each guide-star, and the sequence was checked to ensure that these circles touched or overlapped all along the relevant portion of the planet's path. (The plates used actually have a field of over 1.5 degrees radius, but it was regarded as safer to keep the planet near to the centre of the plate - partly to make the measuring more accurate, and partly to guard against the possibility of the planet's true position being somewhat different from that tabulated.) Where possible, the guide-stars were chosen to have apparent magnitude between 6 and 8.5.

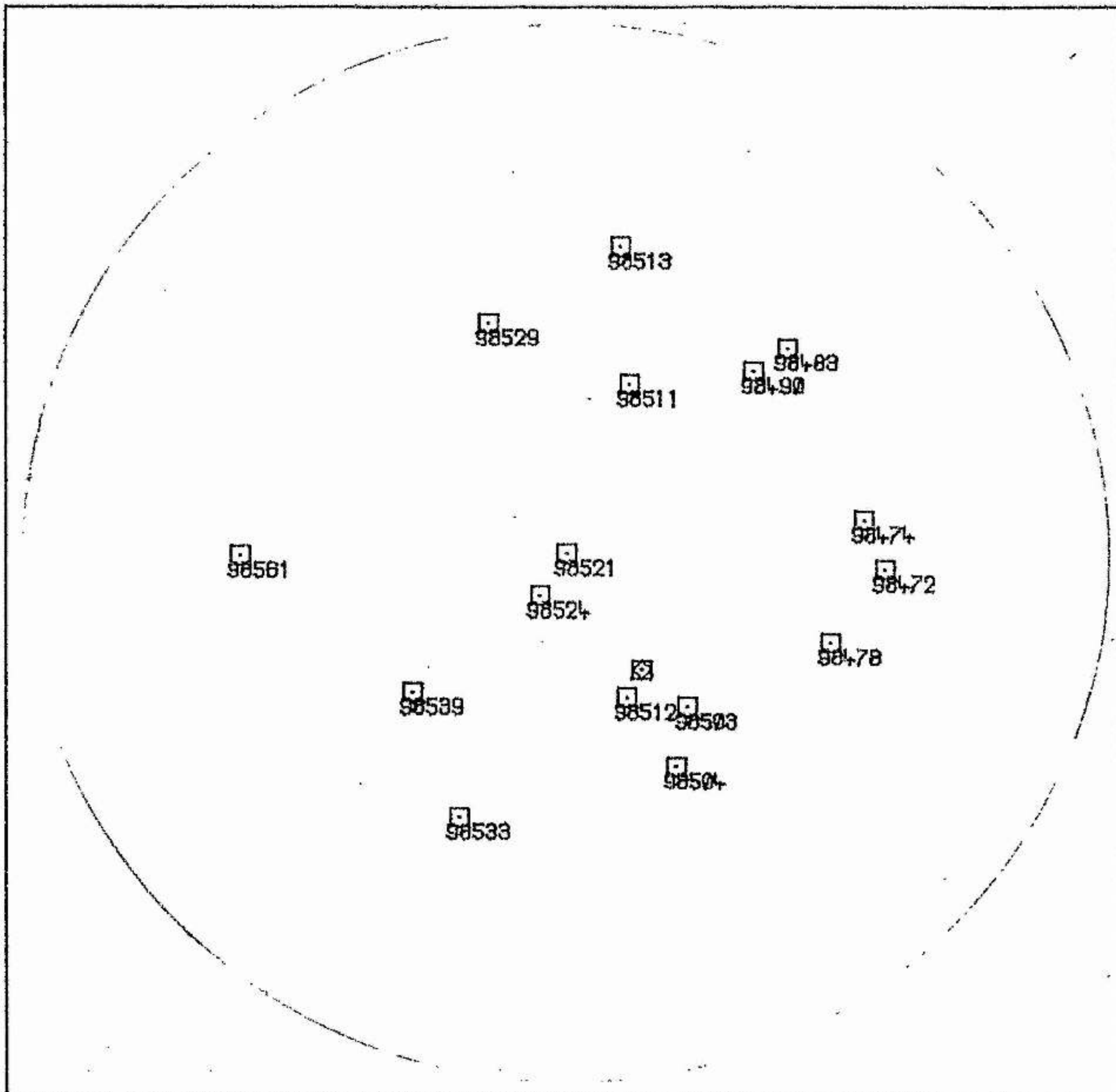
The guide-stars chosen on the chart were identified by number in the SAO catalogue, and listed, together with their positions (corrected for precession) and the dates during which the minor planet would be within 0.75 degrees of them. A finding-chart was also prepared for each guide-star. At first this was done by hand, simply by copying and enlarging, free-hand, a small area of the relevant SAO Star Atlas page. Later, charts were prepared by computer, as follows.

All the stars which appeared, from inspection of the Star Atlas, to be within 1.5 degrees of the guide-star were listed. Their SAO numbers, positions and proper motions were fed into the computer, together with an unnumbered "star" which represented the approximate position of the minor planet (obtained by interpolation from the "Ephemerides of Minor Planets"). A simple computer program then calculated the standard coordinates of each star, according to the usual formulae:

$$x = K \frac{\cos(\delta)\sin(\alpha-\alpha_0)}{\sin(\delta)\sin(\delta_0) + \cos(\delta)\cos(\delta_0)\cos(\alpha-\alpha_0)}$$

$$y = K \frac{\sin(\delta)\cos(\delta_0) - \cos(\delta)\sin(\delta_0)\cos(\alpha-\alpha_0)}{\sin(\delta)\sin(\delta_0) + \cos(\delta)\cos(\delta_0)\cos(\alpha-\alpha_0)}$$

where (α,δ) are the coordinates of the star, (α_0,δ_0) the coordinates of the guide-star, and K a constant to convert the standard coordinates from radians to millimetres. The value used for K was 2572.2, corresponding to a conversion factor of 80.19"/mm. (See appendix 1.)



COMPUTER-DRAWN STAR-CHART

for plate 43

Diagram 3

The computer then plotted these coordinates to give a chart of all the stars, each star being labelled with its SAO number, and a different symbol being used to mark the approximate position of the minor planet. (See diagram 3, which is the finding-chart for plate no. 43.) The first use of this chart was as a finding-chart, to assist in locating and identifying the correct guide-star through the telescope finder, when a plate was about to be taken. Later, when the plate had been taken and developed, the chart was used again to help in identifying all the known stars on it (i.e. those in the SAO catalogue), and to indicate the approximate position of the minor planet. Finally, the table of standard coordinates, generated by the computer, was used when the plate was to be measured.

The next step was to decide how many plates were to be taken, and at what intervals. Once photographed, the minor planet could only be identified by the fact that its position changed over a period of time; therefore at least two plates were needed, to be compared. A blink comparator microscope was used for this, and it was found that the smallest detectable position change occurred over an hour or so, and that the largest movement easily recognised corresponded to about two days. (The first condition is set by the size of the image on the plate: if the apparent movement is less than the diameter of the image it is not easy to see. The second condition is set by the size of field of the eye-piece of the blink comparator: if the two images cannot be brought into the same field of view, again the movement is not easy to see.)

It was planned therefore to take two plates of each minor planet on the first night of the fortnight, separated by an hour or so, and then to take subsequent plates at two-day intervals for as long as the same guide-star could be used. When the minor planet moved too far from that guide-star and it was necessary to transfer to the next in the sequence, two plates would be taken on the same night in rapid succession, one on each guide-star. The minor planet could be found on the first plate (on the old guide-star) by comparison with the previous plate in the sequence. The position of the minor planet on the second plate (on the new guide-star), could then be found by overlapping the plates so that their common areas were superposed. Thereafter plates would be taken on the new guide-star at two-day intervals, and the whole process would be repeated as often as necessary until the end of the fortnight.

In practice, of course, the plan could not be followed closely. The main cause of disruption was the weather, which often prevented plates being taken at two-day intervals; it was then necessary to take two plates on the next clear night, to start the sequence again. It also happened occasionally that a plate was accidentally taken on the wrong guide-star, thus spoiling the sequence. If it was not too far from the right guide-star, it was sometimes still possible to compare them on the blink comparator. But in other cases another plate had to be taken on the erroneous guide-star some days or weeks later, in order to locate the minor planet on the first plate (by its apparent appearance and disappearance in the blink comparator, instead of by its apparent movement.) In a few cases, the minor planet was located on an isolated plate by the laborious process of comparing the plate with the corresponding area of the sky in the photographic prints of the Palomar Sky Survey. Table 1 shows all the plates that proved usable, for each of the minor planets observed, together with the SAO numbers and coordinates of their guide-stars.

All but two of the plates used were 6-inch diameter circular plates coated with Ilford SRO emulsion. Exposure times were generally 30 minutes during the first winter, but it was found that shorter exposures gave a clearer, rounder image for the minor planet without materially affecting the quality of the images of the stars that were to be measured as reference stars; by the third winter, exposure times had generally been reduced to 10-15 minutes. In a few cases, the sky clouded over before an exposure was complete. When this happened, the shutter over the plate was closed, but the telescope was allowed to continue driving for some time. If the weather remained bad, the exposure was eventually abandoned and the plate developed anyway. But if the sky cleared the guide-star was centred in the field and the shutter was re-opened, until the total exposure time was completed. On such a plate, the star images appeared normal, but the image of the minor planet was abnormally lengthened, and sometimes actually separable into two parts, owing to the planet having moved while the shutter was closed. Where two images were visible, they were subsequently measured separately and treated as two independent observations. A single elongated image was measured by centring it as accurately as possible in the measuring machine and taking the time of the observation as the weighted mean of the times of the separate exposures.

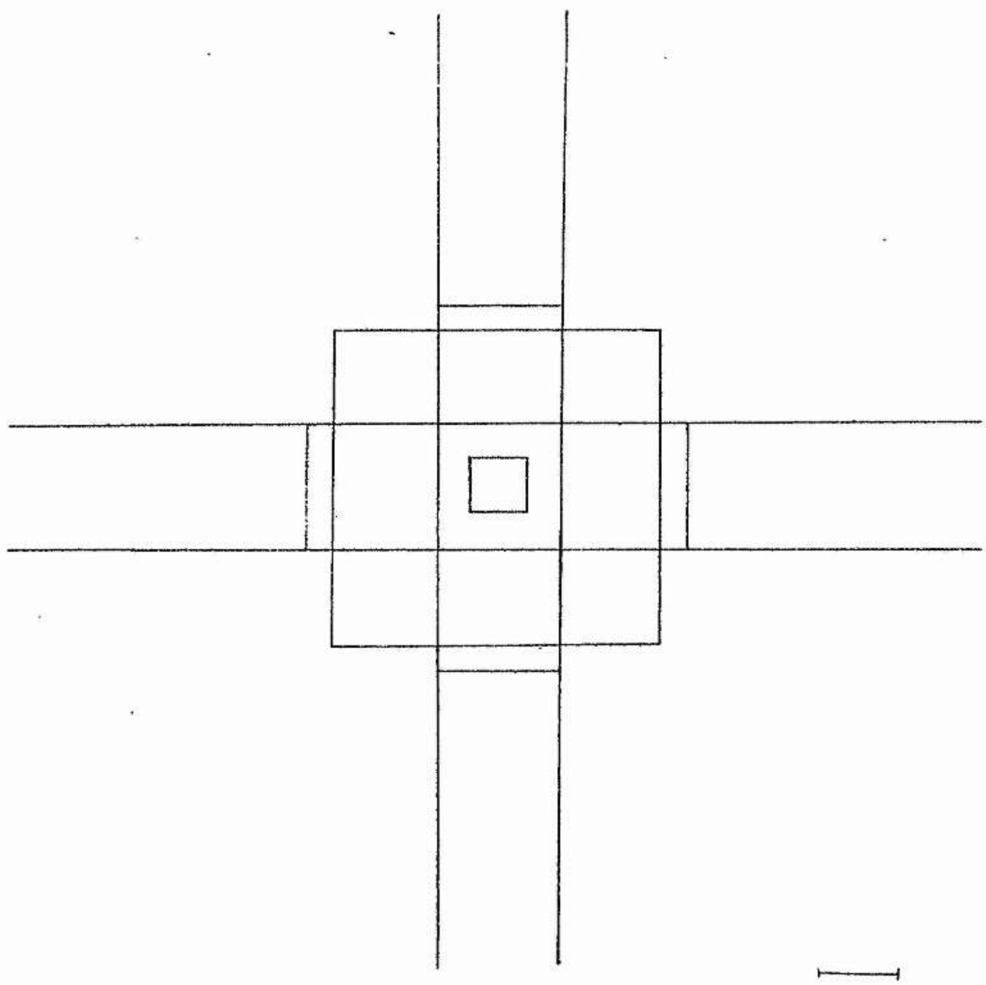
The plates taken on a given night were generally developed the following day, and examined within a day or two. Each plate was placed on top of the computer-drawn chart described above, and moved until the images of the brightest stars coincided with the positions of the catalogued stars plotted on the chart. Each star-image corresponding to a catalogued star was then marked with a spot of green ink on the glass side of the plate, the guide-star in the centre being distinguished by having two green spots. A circle of red ink was drawn around the approximate position of the minor planet, to limit the area of the plate that was to be searched on the blink comparator.

The blink comparator microscope is a standard model produced by Grubb Parsons Ltd. of Newcastle-upon-Tyne. The two plate-holders are set on a heavy horizontal carriage, which can be moved in two dimensions by control knobs, the x- and y-position of the carriage being registered on millimetre scales. One plate-holder can be moved relative to the other by small distances in x and y, and by a considerable angle in orientation. The former adjustment was occasionally useful when the two plates to be compared were taken on different, but close, guide-stars. The latter adjustment was always very useful with the circular plates: although each plate shows distinct marks at its outer edge from the flanges of the telescope plate-holder, these are only a rough guide to orienting it correctly in any measuring-machine.

A small area of each plate, depending on the position of the carriage, is illuminated, by a separate lamp for each plate. The images of both plates are brought together to a binocular eye-piece at the front of the machine. The associated electronic controls enable each lamp to be switched on or off manually, or both lamps to be "blinked" - i.e. switched on and off alternately automatically. The rate of "blinking" can be varied to suit the operator.

In practice, both lamps were switched on manually while the plates were set up and adjusted. The movable plate-holder was adjusted in x and y to make the two images of the centre (guide-)star coincide; the carriage was then moved to bring an area near the edge of the plates under examination, and the movable plate-holder was adjusted in orientation until the images of the stars in that area coincided. As the guide-star was not necessarily at the exact centre of the plate, this adjustment in orientation generally caused a slight displacement of the images of the guide-star, so the two processes had to be repeated once or twice, until the star-images coincided in all parts of the plate. Then the lamps were set to "blink" at a comfortable rate, and the area circled in red ink was examined. Usually it took only a few minutes to pick out the image of the minor planet by its apparent movement to and fro. The ink circle was then wiped off each plate and replaced by a spot of red ink marking the exact position of the minor planet. The plates were then ready to be measured.

All the plates were examined and "blinked" as soon as possible, so that, if any one proved unusable, another plate could be taken straight away. The measuring, however, was a much longer process, and was generally carried out some months later - chiefly during the summer months, when no observing was being done.



0.1 mm.

(approximately equivalent to 8" arc.)

ZEISS MEASURING-MACHINE GRATICULE

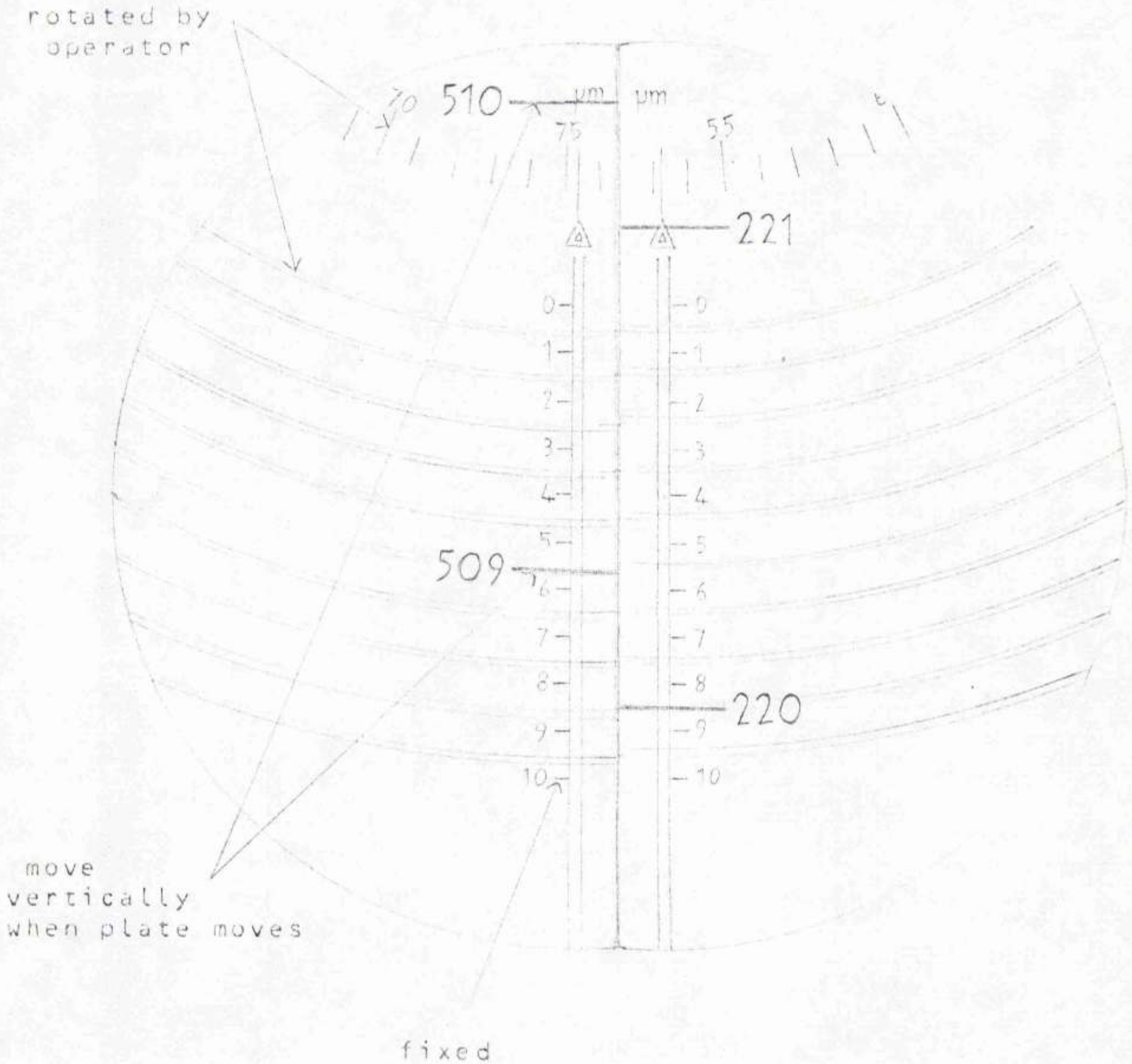
Diagram 4

The plates were measured on a two-coordinate measuring-machine made by Carl Zeiss of Jena. (A photograph of an identical machine appears in Roth 1962.) A heavy metal table, supported on a concrete floor, carries an optically flat glass surface, on which the plate-holder moves freely in any direction. A lamp illuminates a small portion of the plate, its image being brought by an optical system to an eyepiece at the front of the machine. The system includes a graticule (see diagram 4), and the plate is moved to bring the image of the star to be measured to the centre of the graticule. Small screws are attached to the large knob which moves the plate-holder, permitting fine adjustments to be made to the position both in x and in y.

This design of graticule gives excellent accuracy for measuring all except the very smallest images, which lie entirely within the centre square, and the very largest, which cover an area larger than the largest square. One series of five plates was taken on the guide-star SAO 097224, of about 5th magnitude, which gave an image too large for accurate measurement. Its position was nevertheless measured as closely as possible, mainly by aligning the diffraction spikes on the image between the outermost lines of the graticule. This position was taken as the first estimate of the centre of the plate; but the star itself was not used in the subsequent reduction process. On any other plates where a star gave an excessively large image, it was ignored.

A lever on the front of the machine, connected to the optical system, permits the field of view in the eye-piece to be rotated by 90 or 180 degrees. This helps to eliminate any personal bias in centring the image in the graticule. Another control allows the intensity of illumination to be changed; a plate taken on a slightly cloudy night tends to have a somewhat darkened background, requiring greater illumination to distinguish the edges of the star-images.

Once the image has been centred in the graticule, the position of the plate is read from two graduated scales, in x and y. The images of these scales are brought to the same binocular eye-piece; a switch allows the operator to select either the plate or the scales. The scales are displayed in such a way as to facilitate very accurate readings (see diagram 5):



ZEISS MEASURING-MACHINE
SCALES

Diagram 5

The field is split, to show the x-scale on the left and the y-scale on the right, against a pale green background to minimise eye-strain. On each side, as the plate is moved, the image of a scale marked in millimetres moves vertically over a fixed scale marked from zero to ten, the total length of the fixed scale being equal to the distance between the millimetre marks. The millimetre mark visible, and its position on the fixed scale, give the position to one decimal place of millimetres.

Further accuracy is obtained by the use of a vernier-like device, a pair of spiral plates, whose images are superimposed one on each side of the field. Each carries a circular scale, divided into 100, a part of which is seen at the top of the field. This is surrounded by a double line which describes a spiral ten times around the centre, with a constant spacing equal to one division on the fixed vertical scale, i.e. equivalent to 0.1 millimetres. Each of these plates is rotated manually by the operator; when the visible segments of the double lines exactly bracket the marks on the vertical scale, then the circular scale reads zero against a fixed marker. When the spiral plate is rotated so that a segment of the double lines brackets the millimetre mark on the moving scale, then the circular scale shows how far it is displaced from the fixed marks on the vertical scale, thus giving a further two places of decimals. By visual interpolation between the divisions on the circular scale, a further decimal place can be estimated, thus giving a reading accurate to 0.0001 millimetre - probably more accurate than the actual centring of the star image in the graticule. (The diagram shows a position which would be read as x = 509.5753, y = 220.8532.)

The procedure adopted for measuring a plate was to work across it from one side to the other, measuring each reference star, and the minor planet, in turn. The list of standard coordinates, generated earlier by the computer, was kept at hand; since the approximate position of the centre of the plate was known with respect to the instrument scales, a little mental arithmetic gave the approximate position of each image to be measured. The correct image could then be picked out by the identifying dot of green ink on the glass. This procedure failed occasionally, when the circular plate was slightly misorientated in the plate-holder; in such cases the stars were sometimes measured in the wrong order.

PLATE: 49 DATE MEASURED: 11/5/73 PAGE: 1

098358	
528.7874	199.6335
841	291
878	310
860	284
868	338
840	278

098371	
413.7396	165.7401
378	340
428	427
395	373
437	436
390	413

098387	
485.3264	186.4743
271	825
308	816
289	780
290	812
283	819

098362	
523.3368	212.0644
371	639
378	665
335	626
378	664
331	640

334 Chicago	
512.7420	195.7430
383	460
432	459
427	414
441	359
452	373
477	349
473	369
470	359
441	418

098388	
495.0014	206.3532
5.0030	510
5.0005	530
4.9979	474
5.0020	472
4.9987	500

FORM for PLATE-MEASUREMENTS

Diagram 6

Once the image had been located, it was centred in the graticule and the position was read from the scales and written down on a specially-designed form (see diagram 6). The plate was then moved slightly, and the field rotated through 180 degrees before the image was re-centred and re-measured. Each image was measured several times - a total of six times in most cases, but occasionally ten, particularly for the minor planet, or for any other especially large, small or mis-shapen image. The number of reference stars on each plate varied, depending on the area of sky photographed, but it was never less than 9; the average was about 18. The measuring process for each plate was carried out as swiftly as possible, to reduce the danger of a change in temperature or humidity possibly causing a slight movement in the emulsion on the plate. It took two or three minutes to complete the measurements on each image, so the total measuring-time varied from around 20 minutes to well over an hour. In fact, it was later ascertained that a measuring session lasting over three hours produced no systematic shifts in position.

Immediately after the measurement was completed, the positions for each image were averaged in x and y, sometimes on a desk calculator, but usually through a simple computer-program. For each image, the averages of the six, or ten, readings in x and in y were taken as the position on the plate. The standard deviations in x and y were also calculated:

$$\sigma_x = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$

and

$$\sigma_y = \sqrt{\frac{\Sigma(y - \bar{y})^2}{n}}$$

where $\bar{x}$ and $\bar{y}$ are the averages of x and y. These standard deviations, which gave a measure of the uncertainty attached to each calculated average position, were used to assign a weight to each reference star in the subsequent calculations, as will be described on page 42.

Every plate was measured twice, the second measurement being made, sometimes a few days, sometimes several months, after the first. The procedure was exactly the same, except that the images were measured in the reverse direction (i.e. the last one first), and occasionally stars near the edge of the plate were omitted on the second measurement. Each set of measurements was reduced separately, yielding two independent values for the position of the minor planet at the time of that observation.

Table 2 lists the reference stars used for each plate.

The standard coordinates of a star or planet on a photographic plate depend only on the celestial coordinates of the star or planet and of the centre of the plate. Thus, if the standard coordinates of a minor planet on a particular plate are known, its celestial coordinates can also be found relative to the celestial coordinates of the centre of that plate (with respect to the same equinox and epoch.)

However, the standard coordinates (x,y) differ from the measured coordinates (X,Y), for several reasons. Firstly, in order to relate the standard coordinates (in radians) to the measured coordinates (in millimetres) it is necessary to know the plate-scale, and this is subject to error. Secondly, the actual centre of the plate may be slightly displaced from the assumed centre. Thirdly, the X- and Y-axes used in measuring the plate may not be exactly parallel to the true x- and y-axes of the plate, if the alignments of the plate in the measuring-machine and the telescope are slightly different (this error is particularly common with circular plates).

In addition to these purely geometrical errors, there are also various physical factors affecting the relationship between the standard and the measured coordinates. The most important of these is probably differential refraction, which shifts the images of stars or planets more in one part of the plate than in another; the refraction is greatest in that part of the field that is nearest the horizon. Imperfections in the optical system of the telescope may also lead to relative shifts of the images in different parts of the plate.

Instead of attempting to deal with each of these factors separately, it is simplest to treat their combined effects on each individual plate. Smart (1965) suggests that all the effects are generally linear combinations of the coordinates. In this investigation, it is assumed that effects up to second order may be present. That is, the standard coordinates (x,y) and the measured coordinates (X,Y) are related by expressions of the form:

$$x - X = A_x X^2 + B_x XY + C_x Y^2 + D_x X + E_x Y + F_x$$
$$\qquad (1)$$
$$y - Y = A_y X^2 + B_y XY + C_y Y^2 + D_y X + E_y Y + F_y$$

where A_x, B_x,... F_x and A_y, B_y,..., F_y are constants, for any given plate. (In fact, it is found that the second-order constants A_x, B_x, C_x, A_y, B_y and C_y are far smaller than the linear constants D_x, E_x, F_x, D_y, E_y and F_y.) These plate constants are calculated from the standard and the measured coordinates of a number of stars on the plate, whose celestial coordinates are known from a star catalogue; these are the "reference stars". A unique solution for the plate constants can be obtained from just six reference stars. However, as described on page 34, at least 9 reference stars were measured on each plate. It is obviously desirable to use the information from all these measurements to calculate a more accurate set of plate constants. For this reason, the "method of least squares" was used.

The principle of this method is described in, for example, Smart (1958). The application in this case, for each individual plate, is as follows.

For any choice of the constants $A_x, \ldots, F_x, A_y, \ldots, F_y$, there will be some error or "residual" when they are substituted into equations (1) for any individual star. That is, for the i'th star,

$$(x_i - X_i) - (A_x X_i^2 + B_x X_i Y_i + C_x Y_i^2 + D_x X_i + E_x Y_i + F_x) = (\varepsilon_1)_i$$
$$(y_i - Y_i) - (A_y X_i^2 + B_y X_i Y_i + C_y Y_i^2 + D_y X_i + E_y Y_i + F_y) = (\varepsilon_2)_i$$

These are called the "equations of condition" for the i'th star. The principle of the "method of least squares" is to choose values for $A_x, \ldots, F_x, A_y, \ldots, F_y$ such that the sums of the squares of the residuals for the n reference stars are as small as possible; i.e.

$$S_1 = \sum_{i=1}^{n} (\varepsilon_1)_i^2 \quad \text{and} \quad S_2 = \sum_{i=1}^{n} (\varepsilon_2)_i^2$$

should be as small as possible.

We find the minima of S_1 and S_2 by making

$$\frac{\partial S_1}{\partial A_x} = \frac{\partial S_1}{\partial B_x} = \ldots = \frac{\partial S_1}{\partial F_x} = 0 \text{ and } \frac{\partial S_2}{\partial A_y} = \frac{\partial S_2}{\partial B_y} = \ldots = \frac{\partial S_2}{\partial F_y} = 0$$

These conditions give us two sets of six equations in six unknowns, calles the "normal equations". Taking the symbol Σ to mean the sum over i from 1 to n, they are:

$$A_x \Sigma X_i^4 + B_x \Sigma X_i^3 Y_i + C_x \Sigma X_i^2 Y_i^2 + D_x \Sigma X_i^3$$
$$+ E_x \Sigma X_i^2 Y_i + F_x \Sigma X_i^2 = \Sigma X_i^2 (x_i - X_i)$$
$$A_x \Sigma X_i^3 Y_i + B_x \Sigma X_i^2 Y_i^2 + C_x \Sigma X_i Y_i^3 + D_x \Sigma X_i^2 Y_i$$
$$+ E_x \Sigma X_i Y_i^2 + F_x \Sigma X_i Y_i = \Sigma X_i Y_i (x_i - X_i)$$
$$A_x \Sigma X_i^2 Y_i^2 + B_x \Sigma X_i Y_i^3 + C_x \Sigma Y_i^4 + D_x \Sigma X_i Y_i^2$$
$$+ E_x \Sigma Y_i^3 + F_x \Sigma Y_i^2 = \Sigma Y_i^2 (x_i - X_i)$$
$$A_x \Sigma X_i^3 + B_x \Sigma X_i^2 Y_i + C_x \Sigma X_i Y_i^2 + D_x \Sigma X_i^2$$
$$+ E_x \Sigma X_i Y_i + F_x \Sigma X_i = \Sigma X_i (x_i - X_i)$$
$$A_x \Sigma X_i^2 Y_i + B_x \Sigma X_i Y_i^2 + C_x \Sigma Y_i^3 + D_x \Sigma X_i Y_i$$
$$+ E_x \Sigma Y_i^2 + F_x \Sigma Y_i = \Sigma Y_i (x_i - X_i)$$
$$A_x \Sigma X_i^2 + B_x \Sigma X_i Y_i + C_x \Sigma Y_i^2 + D_x \Sigma X_i$$
$$+ E_x \Sigma Y_i + F_x n = \Sigma (x_i - X_i)$$

and

$$A_y \Sigma X_i^4 + B_y \Sigma X_i^3 Y_i + C_y \Sigma X_i^2 Y_i^2 + D_y \Sigma X_i^3$$
$$+ E_y \Sigma X_i^2 Y_i + F_y \Sigma X_i^2 = \Sigma X_i^2 (y_i - Y_i)$$
$$A_y \Sigma X_i^3 Y_i + B_y \Sigma X_i^2 Y_i^2 + C_y \Sigma X_i Y_i^3 + D_y \Sigma X_i^2 Y_i$$
$$+ E_y \Sigma X_i Y_i^2 + F_y \Sigma X_i Y_i = \Sigma X_i Y_i (y_i - Y_i)$$
$$A_y \Sigma X_i^2 Y_i^2 + B_y \Sigma X_i Y_i^3 + C_y \Sigma Y_i^4 + D_y \Sigma X_i Y_i^2$$
$$+ E_y \Sigma Y_i^3 + F_y \Sigma Y_i^2 = \Sigma Y_i^2 (y_i - Y_i)$$
$$A_y \Sigma X_i^3 + B_y \Sigma X_i^2 Y_i + C_y \Sigma X_i Y_i^2 + D_y \Sigma X_i^2$$
$$+ E_y \Sigma X_i Y_i + F_y \Sigma X_i = \Sigma X_i (y_i - Y_i)$$
$$A_y \Sigma X_i^2 Y_i + B_y \Sigma X_i Y_i^2 + C_y \Sigma Y_i^3 + D_y \Sigma X_i Y_i$$
$$+ E_y \Sigma Y_i^2 + F_y \Sigma Y_i = \Sigma Y_i (y_i - Y_i)$$
$$A_y \Sigma X_i^2 + B_y \Sigma X_i Y_i + C_y \Sigma Y_i^2 + D_y \Sigma X_i$$
$$+ E_y \Sigma Y_i + F_y n = \Sigma (y_i - Y_i)$$

The solution of these two sets of equations yields values of $A_x, \ldots, F_x, A_y, \ldots, F_y$ which are taken as the plate constants for the particular plate in question. They are used to convert the measured coordinates of the minor planet into standard coordinates, from which its celestial coordinates can then be found.

The computer program written for these calculations incorporated certain features in addition to the basic procedure outlined above. To begin with, the reference stars were not all given equal weight in the calculations. As described on page 35, some star-images could be measured more accurately than others; this was reflected in the fact that the repeated measurements agreed more closely, and hence that the standard deviation from the mean position was less. A weighting system was therefore adopted that would give least weight to the stars with the largest standard deviation. (The standard deviations in x and y were treated separately, and each star was given two independent weights, one for x, one for y.)

Inspection showed that the average standard deviation was about 0.0007 mm., so stars with this standard deviation were given a weight of 1. The weights for the other stars were calculated by an inverse square formula: for standard deviation σ, the weight was $(0.0007/\sigma)^2$. This ensured that very low weights were given to the least accurately-measured stars.

The i'th star was thus given weights $(w_x)_i$ and $(w_y)_i$ for its x- and y-coordinates. This leads to a modification of the "normal equations" given above, since each term for the i'th star must be multiplied by the appropriate weight $(w_x)_i$ or $(w_y)_i$ before being summed. Thus the normal equations become:

$$A_x \Sigma (w_x)_i X_i^4 + B_x \Sigma (w_x)_i X_i^3 Y_i + C_x \Sigma (w_x)_i X_i^2 Y_i^2$$
$$+ D_x \Sigma (w_x)_i X_i^3 + E_x \Sigma (w_x)_i X_i^2 Y_i + F_x \Sigma (w_x)_i X_i^2$$
$$= \Sigma (w_x) X_i^2 (x_i - X_i)$$

$$A_x \Sigma (w_x) X_i^3 Y_i + B_x \Sigma (w_x)_i X_i^2 Y_i^2 + C_x \Sigma (w_x)_i X_i Y_i^3$$
$$+ D_x \Sigma (w_x)_i X_i^2 Y_i + E_x \Sigma (w_x)_i X_i Y_i^2 + F_x \Sigma (w_x)_i X_i Y_i$$
$$= \Sigma (w_x)_i X_i Y_i (x_i - X_i)$$

$$A_x \Sigma (w_x)_i X_i^2 Y_i^2 + B_x \Sigma (w_x)_i X_i Y_i^3 + C_x \Sigma (w_x)_i Y_i^4$$
$$+ D_x \Sigma (w_x) X_i Y_i^2 + E_x \Sigma (w_x) Y_i^3 + F_x \Sigma (w_x) Y_i^2$$
$$= \Sigma (w_x)_i Y_i^2 (x_i - X_i)$$

$$A_x \Sigma (w_x)_i X_i^3 + B_x \Sigma (w_x)_i X_i^2 Y_i + C_x \Sigma (w_x)_i X_i Y_i^2$$
$$+ D_x \Sigma (w_x)_i X_i^2 + E_x \Sigma (w_x)_i X_i Y_i + F_x \Sigma (w_x)_i X_i$$
$$= \Sigma (w_x)_i X_i (x_i - X_i)$$

$$A_x \Sigma (w_x)_i X_i^2 Y_i + B_x \Sigma (w_x)_i X_i Y_i^2 + C_x \Sigma (w_x)_i Y_i^3$$
$$+ D_x \Sigma (w_x)_i X_i Y_i + E_x \Sigma (w_x)_i Y_i^2 + F_x \Sigma (w_x)_i Y_i$$
$$= \Sigma (w_x)_i Y_i (x_i - X_i)$$

$$A_x \Sigma (w_x)_i X_i^2 + B_x \Sigma (w_x)_i X_i Y_i + C_x \Sigma (w_x)_i Y_i^2$$
$$+ D_x \Sigma (w_x)_i X_i + E_x \Sigma (w_x)_i Y_i + F_x \Sigma (w_x)_i$$
$$= \Sigma (w_x)_i (x_i - X_i)$$

and

$$A_y \sum (w_y)_i X_i^4 + B_y \sum (w_y)_i X_i^3 Y_i + C_y \sum (w_y)_i X_i^2 Y_i^2$$
$$+ D_y \sum (w_y)_i X_i^3 + E_y \sum (w_y)_i X_i^2 Y_i + F_y \sum (w_y)_i X_i^2$$
$$= \sum (w_y)_i X_i^2 (y_i - Y_i)$$

$$A_y \sum (w_y)_i X_i^3 Y_i + B_y \sum (w_y) X_i^2 Y_i^2 + C_y \sum (w_y)_i X_i Y_i^3$$
$$+ D_y \sum (w_y)_i X_i^2 Y_i + E_y \sum (w_y)_i X_i Y_i^2 + F_y \sum (w_y)_i X_i Y_i$$
$$= \sum (w_y)_i X_i Y_i (y_i - Y_i)$$

$$A_y \sum (w_y)_i X_i^2 Y_i^2 + B_y \sum (w_y)_i X_i Y_i^3 + C_y \sum (w_y)_i Y_i^4$$
$$+ D_y \sum (w_y)_i X_i Y_i^2 + E_y \sum (w_y)_i Y_i^3 + F_y \sum (w_y)_i Y_i^2$$
$$= \sum (w_y)_i Y_i^2 (y_i - Y_i)$$

$$A_y \sum (w_y)_i X_i^3 + B_y \sum (w_y)_i X_i^2 Y_i + C_y \sum (w_y)_i X_i Y_i^2$$
$$+ D_y \sum (w_y)_i X_i^2 + E_y \sum (w_y)_i X_i Y_i + F_y \sum (w_y)_i X_i$$
$$= \sum (w_y)_i X_i (y_i - Y_i)$$

$$A_y \sum (w_y)_i X_i^2 Y_i + B_y \sum (w_y)_i X_i Y_i^2 + C_y \sum (w_y)_i Y_i^3$$
$$+ D_y \sum (w_y)_i X_i Y_i + E_y \sum (w_y)_i Y_i^2 + F_y \sum (w_y)_i Y_i$$
$$= \sum (w_y)_i Y_i (y_i - Y_i)$$

$$A_y \sum (w_y)_i X_i^2 + B_y \sum (w_y)_i X_i Y_i + C_y \sum (w_y)_i Y_i^2$$
$$+ D_y \sum (w_y)_i X_i + E_y \sum (w_y)_i Y_i + F_y \sum (w_y)_i$$
$$= \sum (w_y)_i (y_i - Y_i)$$

The computer program began by reading in the catalogued positions of the reference stars. These were written in the same format as for the earlier program which calculated and plotted standard coordinates; data was input by means of punched cards, one star to a card, so this meant that the same cards could be used for each program. (A total of 535 different reference stars was used.) The program then read in the measured positions of the same reference stars, followed by the measured position of the minor planet. Each position was accompanied by its "errors", the standard deviations in the average positions of x and y; these were used to calculate the weights in x and y, w_x and w_y, for each star, as described above. The program then calculated the standard coordinates from the celestial coordinates by the usual formulae

$$x = \frac{\cos(\delta)\ \sin(\alpha-\alpha_0)}{\sin(\delta)\ \sin(\delta_0) + \cos(\delta)\ \cos(\delta_0)\ \cos(\alpha-\alpha_0)}$$

$$y = \frac{\sin(\delta)\ \cos(\delta_0) - \cos(\delta)\ \sin(\delta_0)\ \cos(\alpha-\alpha_0)}{\sin(\delta)\ \sin(\delta_0) + \cos(\delta)\ \cos(\delta_0)\ \cos(\alpha-\alpha_0)}$$

where (α_0, δ_0) are the coordinates of the centre of the plate. It then used the standard and the measured coordinates to produce two sets of six equations in six unknowns, as described above, from which the twelve plate constants could be calculated.

Each set of equations was solved by reducing the matrix of coefficients to an upper triangular matrix. That is, the first equation was divided by its first coefficient, and then appropriate multiples of the first equation were subtracted from the following equations to make their first coefficients zero. The procedure was repeated with the second coefficient of the second equation, and so on until all the coefficients below the diagonal were zero.

The sixth equation now gave the sixth unknown directly; this was substituted into the fifth equation to give the fifth unknown, these two gave the fourth unknown, and so on. Thus each set of six plate constants was determined.

The program also calculated the errors in these values of the plate constants, according to the following method.

Consider one set of normal equations:

$$A\sum X^4 + B\sum X^3Y + C\sum X^2Y^2 + D\sum X^3 + E\sum X^2Y + F\sum X^2 = \sum X^2r$$
$$A\sum X^3Y + B\sum X^2Y^2 + C\sum XY^3 + D\sum X^2Y + E\sum XY^2 + F\sum XY = \sum XYr$$
$$A\sum X^2Y^2 + B\sum XY^3 + C\sum Y^4 + D\sum XY^2 + E\sum Y^3 + F\sum Y^2 = \sum Y^2r$$
$$A\sum X^3 + B\sum X^2Y + C\sum XY^2 + D\sum X^2 + E\sum XY + F\sum X = \sum Xr$$
$$A\sum X^2Y + B\sum XY^2 + C\sum Y^3 + D\sum XY + E\sum Y^2 + F\sum Y = \sum Yr$$
$$A\sum X^2 + B\sum XY + C\sum Y^2 + D\sum X + E\sum Y + F\,n = \sum r$$

where: $\sum q$ means $\sum_{i=1}^{n} q_i$ for any quantity q; A,..., F are the plate constants in either x or y (i.e. either A_x,..., F_x or A_y,..., F_y); and r_i are the corresponding residuals in either x or y (i.e. either $x_i - X_i$ or $y_i - Y_i$).

The solution of these equations gives A,...,F as functions of the coordinates (X_i, Y_i) and of the residuals r_i. We may write these solutions as linear combinations of the residuals

$A = a_1 r_1 + a_2 r_2 + \ldots + a_6 r_6$

$B = b_1 r_1 + b_2 r_2 + \ldots + b_6 r_6$

and so on to

$F = f_1 r_1 + f_2 r_2 + \ldots + f_6 r_6$

where the coefficients $a_1,\ldots,a_6,\ldots\ldots,f_1,\ldots,f_6$ are (unknown) functions of the coordinates (X_i, Y_i).

The mean square error of a single observation is given by

$$\mu = \sqrt{\frac{\sum r^2}{n - 6}}$$

So the errors in the plate constants are

$dA^2 = a_1^2 \mu^2 + a_2^2 \mu^2 + \ldots + a_6^2 \mu^2$

$dB^2 = b_1^2 \mu^2 + b_2^2 \mu^2 + \ldots + b_6^2 \mu^2$

and so on to

$dF^2 = f_1^2 \mu^2 + f_2^2 \mu^2 + \ldots + f_6^2 \mu^2$

Thus it is necessary to determine the functions $\sum a^2, \sum b^2, \ldots, \sum f^2$.

We take these new expressions for $A, \ldots, F$ and substitute them into the normal equations:

$$\sum ar \sum X^4 + \sum br \sum X^3 Y + \sum cr \sum X^2 Y^2 + \sum dr \sum X^3 + \sum er \sum X^2 Y + \sum fr \sum X^2 = \sum X^2 r$$

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$$\sum ar \sum X^2 + \sum br \sum XY + \sum cr \sum Y^2 + \sum dr \sum X + \sum er \sum Y + \sum fr\; n = \sum r$$

Collecting together all terms in r_i:

$$a_i \sum X^4 + b_i \sum X^3 Y + c_i \sum X^2 Y^2 + d_i \sum X^3 + e_i \sum X^2 Y + f_i \sum X^2 = X_i^2$$

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$$a_i \sum X^2 + b_i \sum XY + c_i \sum Y^2 + d_i \sum X + e_i \sum Y + f_i\, n = 1$$

(*) We now multiply each equation by a_i:

$$a_i^2 \sum X^4 + a_i b_i \sum X^3 Y + a_i c_i \sum X^2 Y^2 + a_i d_i \sum X^3 + a_i e_i \sum X^2 Y + a_i f_i \sum X^2 = a_i \; X_i^2$$

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$$a_i^2 \sum X^2 + a_i b_i \sum XY + a_i c_i \sum Y^2 + a_i d_i \sum X + a_i e_i \sum Y + a_i f_i\, n = a_i$$

If we now sum each equation over i from 1 to n, we have:

$$\sum a^2 \sum X^4 + \sum ab \sum X^3 Y + \sum ac \sum X^2 Y^2 + \sum ad \sum X^3 + \sum ae \sum X^2 Y + \sum af \sum X^2 = \sum a X^2$$

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$$\sum a^2 \sum X^2 + \sum ab \sum XY + \sum ac \sum Y^2 + \sum ad \sum X + \sum ae \sum Y + \sum af\; n = \sum a$$

This gives a set of six equations in six unknowns (Σa^2, Σab, Σac, Σad, Σae and Σaf) of which the coefficients on the left-hand side are the same as the coefficients of the normal equations. However, it is necessary to determine the quantities on the right-hand side of these equations. To do this, we return to the original equation of condition for the i'th star:

$$AX_i^2 + BX_iY_i + CY_i^2 + DX_i + EY_i + F = r_i$$

Multiplying through by a_i gives

$$a_iAX_i^2 + a_iBX_iY_i + a_iCY_i^2 + a_iDX_i + a_iEY_i + a_iF = a_ir_i$$

Rearranging and summing over i gives

$$A\ \Sigma aX^2 + B\ \Sigma aXY + C\ \Sigma aY^2 + D\ \Sigma aX + E\ \Sigma aY + F\ \Sigma a = \Sigma ar$$

But $\Sigma ar = A$ (by definition)

So $\Sigma aX^2 = 1$

$$\Sigma aXY = \Sigma aY^2 = \Sigma aX = \Sigma aY = \Sigma a = 0$$

So all the coefficients for the six equations in $\Sigma a^2, \ldots, \Sigma af$ are known, and the equations can be solved.

Now, returning to the point (*), instead of multiplying each equation by a_i we can multiply instead by b_i. Then, when we sum each equation over i, we obtain:

$$\Sigma ab\ \Sigma X^4 + \Sigma b^2\ \Sigma X^3Y + \Sigma bc\ \Sigma X^2Y^2 + \Sigma bd\ \Sigma X^3$$
$$+ \Sigma be\ \Sigma X^2Y + \Sigma bf \Sigma X^2 = \Sigma b\ X^2$$

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$$\Sigma ab\ \Sigma X^2 + \Sigma b^2 \Sigma X\ Y + \Sigma bc \Sigma Y^2 + \Sigma bd\ \Sigma X$$
$$+ \Sigma be\ \Sigma Y + \Sigma bf\ n = \Sigma b$$

Once again, we have six equations in six unknowns, this time for $\sum ab$, $\sum b^2$, $\sum bc$, $\sum bd$, $\sum be$ and $\sum bf$. The coefficients on the left-hand side are still the same as in the normal equations, but once again we have to determine the terms on the right-hand side. Multiplying the equation of condition for the i'th star by b_i gives

$$b_i AX_i^2 + b_i BX_i Y_i + b_i CY_i^2 + b_i DX_i + b_i EY_i + b_i F_i = b_i r_i$$

and summing gives

$$A \sum bX^2 + B \sum bXY + C \sum bY^2 + D \sum bX + E \sum bY + F \sum b = \sum br = B$$

so $\sum bXY = 1$

$$\sum bX^2 = \sum bY^2 = \sum bX = \sum bY = \sum b = 0$$

So these equations can also be solved, for $\sum ab$, $\sum b^2$, $\sum bc$, $\sum bd$, $\sum be$ and $\sum bf$.

In the same way, four more sets of equations can be produced and solved, giving values for $\sum pq$ where p and q are any of a,...,f. In particular, the quantities $\sum a^2,\ldots,\sum f^2$ are now known, and consequently the errors dA,...,dF can be calculated.

So the computer program calculated the two sets of plate constants A_x, B_x,...., F_x, A_y, B_y,....,F_y, and their associated errors dA_x, dB_x,...., dF_x, dA_y, dB_y,...., dF_y, from the reference stars.

It then used these to calculate the standard coordinates of the minor planet, with their associated errors.

The formulae for the standard coordinates (x,y) are simply

$$x = A_x X^2 + B_x XY + C_x Y^2 + D_x X + E_x Y + F_x + X$$

$$y = A_y X^2 + B_y XY + C_y Y^2 + D_y X + E_y Y + F_y + Y$$

The errors dx and dy in the standard coordinates x and y depend on the errors dA_x,....,dF_x, dA_y,....,dF_y in the plate constants, and also on the errors dX and dY in the measured coordinates (X,Y). Including the error terms in the equation above for x, we have

$$\begin{aligned}
(x+dx)-(X+dX) &= (A_x+dA_x)(X+dx)^2 + (B_x+dB_x)(X+dX)(Y+dY) \\
&\quad + (C_x+dC_x)(Y+dY)^2 + (D_x+dD_x)(X+dX) \\
&\quad + (E_x+dE_x)(Y+dY) + (F_x+dF_x) \\
&= A_x X^2 + B_x XY + C_x Y^2 + D_x X + E_x Y + F_x \\
&\quad + dA_x X^2 + dB_x XY + dC_x Y^2 \\
&\quad + dD_x X + dE_x Y + dF_x \\
&\quad + (2A_x X + B_x Y + D_x)dX + (B_x X + 2C_x Y + E_x)dY
\end{aligned}$$

ignoring terms containing powers or multiples of error terms. So:

$$dx = dA_x X^2 + dB_x XY + dC_x Y^2 + dD_x X + dE_x Y + dF_x$$
$$+ (2A_x X + B_x Y + D_x + 1)\ dX + (B_x X + 2C_x Y + E_x)\ dY$$

Similarly it can be shown that

$$dy = dA_y X^2 + dB_y XY + dC_y Y^2 + dD_y X + dE_y Y + dF_y$$
$$+ (2A_y X + B_y Y + D_y)\ dX + (B_y X + 2C_y Y + E_y + 1)\ dY$$

Now the computer program took these standard coordinates (x,y) with their errors dx and dy, and from them calculated the celestial coordinates (α,δ) of the minor planet, with their associated errors $d\alpha$ and $d\delta$.

The expressions for calculating α and δ from x and y are the reverse of the formulae given on page 19 for the standard coordinates; they are:

$$\tan(\alpha-\alpha_o) = \frac{x}{\cos(\delta_o) - y\sin(\delta_o)}$$

$$\sin(\delta) = \frac{\sin(\delta_o) + y\cos(\delta_o)}{\sqrt{1 + x^2 + y^2}}$$

where (α_o,δ_o) are the coordinates of the centre of the plate.

It is now necessary to calculate the errors in (α,δ) in terms of the errors in (x,y). In the case of $d\alpha$, we note first that $(\alpha-\alpha_o)$ is a small angle, and consequently that $\tan(\alpha-\alpha_o)$ is approximately equal to $(\alpha-\alpha_o)$, i.e.

$$\alpha-\alpha_o = x/(\cos(\delta_o) - y\sin(\delta_o))$$

We have $d\alpha/dx = 1/(\cos(\delta_o) - y\sin(\delta_o))$

and $d\alpha/dy = x\sin(\delta_o) / (\cos(\delta_o) - y\sin(\delta_o))^2$

Now $d\alpha^2 = (d\alpha/dx)^2\, dx^2 + (d\alpha/dy)^2\, dy^2$

so

$$d\alpha = \sqrt{\frac{dx^2}{(\cos(\delta_o) - y\sin(\delta_o))^2} + \frac{x^2\sin^2(\delta_o)\, dy^2}{(\cos(\delta_o) - y\sin(\delta_o))^2}}$$

To calculate $d\delta$, we have

$$\sin(\delta) = \frac{\sin(\delta_o) + y\cos(\delta_o)}{\sqrt{1 + x^2 + y^2}}$$

and consequently

$$\cos(\delta) = \sqrt{\frac{(\cos(\delta_o) - y\sin(\delta_o))^2 + x^2}{1 + x^2 + y^2}}$$

Now

$$\cos(\delta)\frac{d\delta}{dx} = \frac{-x(\sin(\delta_o) + y\cos(\delta_o))}{(1 + x^2 + y^2)^{3/2}}$$

so

$$\frac{d\delta}{dx} = \frac{-x(\sin(\delta_o) + y\cos(\delta_o))}{(1 + x^2 + y^2)\sqrt{(\cos(\delta_o) - y\sin(\delta_o))^2 + x^2}}$$

Similarly we have

$$\frac{d\delta}{dy} = \frac{\cos(\delta_o)(1 + x^2) - y\sin(\delta_o)}{(1 + x^2 + y^2)\sqrt{(\cos(\delta_o) - y\sin(\delta_o))^2 + x^2}}$$

so

$$d\delta = \frac{\sqrt{x^2(\sin(\delta_o) + y\cos(\delta_o))^2 dx^2 + (\cos(\delta_o)(1+x^2) - y\sin(\delta_o))^2 dy}}{(1 + x^2 + y^2)\sqrt{(\cos(\delta_o) - y\sin(\delta_o))^2 + x^2}}$$

These expressions for $d\alpha$ and $d\delta$ are in radians; the program converted them to seconds of time and seconds of arc respectively.

Thus the program was able to calculate the celestial coordinates (α, δ) of the minor planet, together with their errors. However, in fact this was not done immediately. As mentioned on pages 37-38, the principal advantage of the method of least squares is that it enables information from more than six star-measurements to be combined. An added advantage is that the accuracy of the results can then be checked on each individual star. (If only six stars had been used to calculate the twelve plate-constants, the results would inevitably have been exactly correct for those six stars, without necessarily being accurate over the rest of the plate.)

Having calculated the plate-constants, therefore, the computer program first applied them to the measured coordinates of each reference star in turn, and calculated its celestial coordinates according to the method described above for the minor planet. These were then compared with the true (catalogued) celestial position of the star, and the error in position was calculated.

Following an idea by Dodd (1972), this information was then used to refine the constants. At the first stage, any star which gave an error of more than 0.01 radians was dropped from the list of reference stars. The entire procedure was then repeated, and a new set of constants calculated. This time any star with an error of more than 0.0001 radians was omitted. After a third calculation, any star with an error of more than 0.000005 radians (about 1"arc) was omitted, and the following calculation produced the final set of plate constants. Then, and only then, were the constants applied to the measured coordinates of the minor planet, and its celestial coordinates calculated.

This procedure ensured against the possibility of any one star, even if measured very accurately (i.e. having a large weight), distorting the plate-constants for some reason (perhaps because imperfections in the telescope optics caused its image to be misplaced). In practice only one or two stars were omitted, if any, usually at the final stage.

At the end, the average weights in x and y of all the remaining stars were calculated, to give an indication of the likely accuracy of the result, independent of the calculated "errors".

As described on page 34, each plate was measured twice, and the entire process of deriving the plate constants and calculating the celestial position of the minor planet was carried out twice. Thus, for each plate, two positions for the minor planet were obtained, each with its own uncertainties in α and δ. The problem then arose of how to combine these into one position, and how to determine the uncertainty in that position. It may happen that the two values of the position each have very small uncertainties, but are rather widely separated; clearly, the combined position must have a rather large uncertainty. Similarly, two values each with large uncertainties, but agreeing closely, give a combined position which is probably more accurate than the uncertainties would indicate. This problem is approached as follows.

We treat each coordinate, α and δ, separately. To simplify the notation, we will call the first value of the coordinate X, with an associated uncertainty of dX, and the second value Y, with an associated uncertainty of dY. We wish to obtain a combined value Z and its uncertainty dZ.

If the values of X and Y were **not** the result of complex calculations, but had been obtained by simply averaging, say, n observations x_i and m observations y_i, then we would have

$$X = \frac{\sum_{i=1}^{n} x_i}{n} \text{ and } Y = \frac{\sum_{i=1}^{m} y_i}{m}$$

The corresponding uncertainties dX and dY would simply be the standard deviations

$$dX^2 = \frac{\sum_{i=1}^{n}(x_i - X)^2}{n} \text{ and } dY^2 = \frac{\sum_{i=1}^{m}(y_i - Y_i)}{m}$$

The combined value Z would then be obtained by averaging all of the n+m observations, and its uncertainty dZ would be the standard deviation of this averaging:

$$Z = \frac{\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i}{n+m}$$

But $\sum_{i=1}^{n} x_i = nX$ and $\sum_{i=1}^{m} y_i = mY$

so $$Z = \frac{nX + mY}{n+m}$$

$$dZ^2 = \frac{\sum_{i=1}^{n}(x_i - Z)^2 + \sum_{i=1}^{m}(y_i - Z)^2}{n+m}$$

so $$(n+m)dZ^2 = \sum_{i=1}^{n} x_i^2 - 2Z\sum_{i=1}^{n} x_i + nZ^2 + \sum_{i=1}^{m} y_i^2 - 2Z\sum_{i=1}^{m} y_i + mZ^2$$

Now $$n\,dX^2 = \sum_{i=1}^{n}(x_i - X)^2 = \sum_{i=1}^{n} x_i^2 - 2X\sum_{i=1}^{n} x_i + nX^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2nX^2 + nX^2$$

$$= \sum_{i=1}^{n} x_i^2 - nX^2$$

so $$\sum_{i=1}^{n} x_i^2 = n(dX^2 + X^2)$$

Similarly

$$\sum_{i=1}^{m} y_i^2 = m(dY^2 + Y^2)$$

$$\text{so } (n+m)dZ^2 = n(dX^2 + X^2) - 2\,\frac{nX+mY}{n+m}\,nX + n\,\frac{(nX+mY)^2}{(n+m)^2}$$

$$+ m(dY^2 + Y^2) - 2\,\frac{nX+mY}{n+m}\,mY + m\,\frac{(nX+mY)^2}{(n+m)^2}$$

$$= n\,dX^2 + m\,dY^2 + nX^2 + mY^2$$

$$-2(nX+mY)\,\frac{nX+mY}{n+m} + (n+m)\,\frac{(nX+mY)^2}{(n+m)^2}$$

$$= n\,dX^2 + m\,dY^2 + nX^2 + mY^2 - \frac{(nX+mY)^2}{n+m}$$

$$= n\,dX^2 + m\,dY^2 + \frac{n^2X^2 + nmX^2 + nmY^2 + m^2Y^2 - (nX+mY)^2}{n+m}$$

$$= n\,dX^2 + m\,dY^2 + \frac{nm(X^2 - 2XY + Y^2)}{n+m}$$

$$= n\,dX^2 + m\,dY^2 + \frac{nm}{n+m}(X-Y)^2$$

So we have

$$dZ^2 = \frac{n\,dX^2 + m\,dY^2}{n+m} + \frac{nm}{(n+m)^2}(X-Y)^2$$

These are the values of Z and dZ that would apply if X were an average of n observations x_i, and Y were an average of m observations y_i.

In fact, since X and Y (the two values for either of the components α or δ) are **not** obtained by simple averaging, we cannot evaluate Z and dZ because we do not have values for n and m. However, we do have weighting factors.

As explained above, each reference star was assigned a weight w in each coordinate x and y. As the computer program calculated and refined the plate constants, it eliminated some of the reference stars. At the end of the program, the average weights of all the remaining reference stars in both coordinates were found. If a position is associated with high average weights, this indicates that reference stars with mainly high weights - that is, with low uncertainties in their measurements - were used to determine it. It therefore seems reasonable to treat the average weights in the two coordinates as weighting factors for the two coordinates of the planet's position.

Returning to our hypothetical situation where the two values for either of the coordinates were found by simple averaging, we might assign weights in this case to be proportional to the number of observations that were averaged. So we would have

$$w_x = K\,n \qquad \text{and} \qquad w_y = K\,m$$

where K is some constant. We can then write

$$Z = \frac{nX+mY}{n+m} = \frac{K^{-1} w_x X + K^{-1} w_y Y}{K^{-1}(w_x + w_y)}$$

$$= \frac{w_x X + w_y Y}{w_x + w_y}$$

$$\text{and } dZ^2 = \frac{n\, dX^2 + m\, dY^2}{n+m} + \frac{nm}{(n+m)^2}(X-Y)^2$$

$$= \frac{K^{-1} w_x dX^2 + K^{-1} w_y dY^2}{K^{-1}(w_x + w_y)} + \frac{K^{-2} w_x w_y}{K^{-2}(w_x+w_y)^2}(X-Y)^2$$

$$= \frac{w_x dX^2 + w_y dY^2}{w_x+w_y} + \frac{w_x w_y}{(w_x+w_y)^2}(X-Y)^2$$

So, by taking the average weight of the reference stars, in each coordinate, to be the weight w in that coordinate, for each of the two observations, we can calculate a combined value for that coordinate, Z, and an associated uncertainty dZ. It will be noticed that the expression for dZ contains two terms, the first being effectively a weighted average of the uncertainties in the two values being combined, and the second depending on the difference between the two values. These means that whether the original values have large uncertainties, or whether they have small uncertainties but do not agree well, the final value has a correspondingly large uncertainty. Only if the two original values have small uncertainties and are also in good agreement, will the final value have a small uncertainty.

Finally, to restate these results in the notation actually used: We have two positions, (α_1, δ_1) and (α_2, δ_2), with their associated uncertainties $d\alpha_1$, $d\delta_1$, $d\alpha_2$ and $d\delta_2$. We also have average weights in x and y for each measurement of the plate, $(w_x)_1$, $(w_y)_1$, $(w_x)_2$ and $(w_y)_2$. We take the weights in x to apply to the coordinates α and those in y to apply to δ. We combine these two results to give a final position (α, δ), with uncertainties $d\alpha$ and $d\delta$ by the formulae:

$$\alpha = \frac{(w_x)_1 \alpha_1 + (w_x)_2 \alpha_2}{(w_x)_1 + (w_x)_2}$$

$$\delta = \frac{(w_y)_1 \delta_1 + (w_y)_2 \delta_2}{(w_y)_1 + (w_y)_2}$$

$$d\alpha^2 = \frac{(w_x)_1 d\alpha_1^2 + (w_x)_2 d\alpha_2^2}{(w_x)_1 + (w_x)_2} + \frac{(w_x)_1 (w_x)_2}{((w_x)_1 + (w_x)_2)^2} (\alpha_1 - \alpha_2)^2$$

$$d = \frac{(w_y)_1 d\delta_1^2 + (w_y)_2 d\delta^2}{(w_y)_1 + (w_y)_2} + \frac{(w_y)_1 (w_y)_2}{((w_y)_1 + (w_y)_2)^2} (\delta_1 - \delta_2)^2$$

The combined results for each plate measured are listed in table 3.

As a check, the measurement and reduction techniques described above were applied to a field of stars whose positions were well known, namely the Pleiades cluster. The details and results of these measurements are found in appendix 2. The calculated positions of the stars were found to be accurate to less than half a second of arc.

The sequence of observations, measurements and reductions described above led to a set of positions for each minor planet, at different times, each with its own associated error. These were now used to determine the orbit for each minor planet; in the case of (909) Ulla and (334) Chicago, two orbits could be determined, for the two successive winters in which they were observed.

For each planet, three positions were chosen, to determine a preliminary orbit, by Gauss' method. One of a series of computer programs developed by Watson (1974) was used to calculate these orbits. Ideally, the three observations chosen for each planet should have been of equal weight (i.e. similar errors) and equally separated in time, preferably by fortnight intervals. This could not be achieved for all the minor planets under investigation; the positions available tended to be grouped together at intervals of about a month (since they were all obtained around new moon). Where possible, three positions were taken from successive months; where a choice of positions was available in any one month, the position with the smallest error was selected.

The positions chosen for the determination of the preliminary orbit of each planet are shown below.

(87) Sylvia

plate	date	U.T.	R.A.	dα	dec.	dδ
99	28/11/73	$01^h\,21^m\,34^s$	$6^h\,37^m\,29^s.582$	$0^s.065$	$+27^\circ\,15'\,34''.00$	$0''.26$
102	24/12/73	00 45 40	6 17 47.956	0.035	+28 34 58.34	0.15
115	17/ 1/74	19 40 46	5 57 50.547	0.031	+29 18 42.81	0.15

(107) Camilla

plate	date	U.T.	R.A.	dα	dec.	dδ
98	28/11/73	00 54 39	6 48 54.400	0.033	+ 9 37 44.04	0.57
107	30/12/73	01 12 16	6 27 57.551	0.023	+ 9 24 39.39	0.15
114	17/ 1/74	19 14 00	6 14 28.535	0.029	+10 4 3.23	0.31

(334) Chicago (1971-1972)

plate	date	U.T.	R.A.	dα	dec.	dδ
19	14/ 1/72	21 41 28	5 38 57.702	0.220	+19 34 37.48	0.61
23a	20/ 1/72	00 17 19	5 36 20.245	0.026	+19 38 52.34	0.78
26	20/ 1/72	23 07 54	5 35 53.624	0.024	+19 39 43.04	0.37

(334) Chicago (1972-1973)

plate	date	U.T.	R.A.	dα	dec.	dδ
43	7/ 2/73	01 20 12	9 17 34.262	0.022	+16 27 5.78	0.10
49	28/ 2/73	03 40 01	9 4 27.367	0.018	+17 40 35.29	0.37
53	8/ 3/73	02 47 37	9 0 32.701	0.041	+18 2 9.73	0.41

(414) Liriope

plate	date	U.T.	R.A.	dα	dec.	dδ
85	27/10/73	00 02 57	4 56 11.274	0.017	+12 53 59.74	0.38
97	28/11/73	00 25 44	4 36 31.690	0.030	+12 30 12.30	0.93
101	24/12/73	00 12 14	4 17 4.768	0.033	+12 52 56.30	0.19

(909) Ulla (1971-1972) .

plate	date	U.T.	R.A.	dα	dec.	dδ
13	22/12/71	02 03 02	8 14 9.880	0.014	+ 6 13 47.85	0.21
25	20/ 1/72	03 24 49	7 55 30.236	0.032	+ 8 12 39.00	0.20
33a	12/ 2/72	21 8 13	7 40 3.416	0.015	+10 38 23.05	0.15

plate	date	U.T.	R.A.	d	dec.	d
(909) Ulla (1972-1973)						
46	10/ 2/73	$00^h 42^m 28^s$	$12^h\ 4^m\ 15^s.173$	$0^s.086$	$+\ 9^\circ 23' 57''.25$	$0''.58$
54	8/ 3/73	03 18 32	11 51 56.584	0.080	+12 39 27.61	0.35
57	24/ 3/73	22 31 18	11 41 38.981	0.013	+14 35 36.75	0.13
(1574) Meyer						
36	8/ 9/72	00 16 8	00 37 33.739	0.109	+21 47 13.90	1.53
38	8/ 9/72	01 29 57	00 37 32.040	0.116	+21 47 7.09	0.71
40	9/ 9/72	23 22 25	00 36 31.729	0.085	+21 43 10.80	0.31

It was clear that the data for (334) Chicago in the first season, and for (1574) Meyer, were likely to be inadequate, as they covered very short time-intervals. (In the case of (1574) Meyer, these were in fact the only three observations obtained.) The data for (334) Chicago in the second season were also likely to give unreliable results, as they were confined to a single month. In the event, no orbit at all could be obtained for (1574) Meyer; both sets of data for (334) Chicago did produce orbits, but they differed rather widely.

The results of the preliminary orbit calculations are shown below.

(87) Sylvia

t = 0h 14 Nov 1973
M = $91\overset{\circ}{.}877565$
ω = $272\overset{\circ}{.}030603$
Ω = $73\overset{\circ}{.}345900$
i = $10\overset{\circ}{.}896489$
e = 0.092936
a = 3.480780

(107) Camilla

t = 0h 14 Nov 1973
M = $346\overset{\circ}{.}030672$
ω = $291\overset{\circ}{.}203544$
Ω = $174\overset{\circ}{.}074511$
i = $9\overset{\circ}{.}947844$
e = 0.073240
a = 3.488974

(334) Chicago (1971-1972)

t = 0h 6 Sep 1971
M = $298\overset{\circ}{.}808481$
ω = $27\overset{\circ}{.}387472$
Ω = $136\overset{\circ}{.}663114$
i = $4\overset{\circ}{.}122108$
e = 0.244288
a = 4.366569

(334) Chicago (1972-1973)

t = 0h 10 Oct 1972
M = $142\overset{\circ}{.}037228$
ω = $204\overset{\circ}{.}810211$
Ω = $131\overset{\circ}{.}137072$
i = $4\overset{\circ}{.}897761$
e = 0.079088
a = 3.814487

(414) Liriope

t = 0h 14 Nov 1973
M = $357\overset{\circ}{.}141942$
ω = $317\overset{\circ}{.}254108$
Ω = $111\overset{\circ}{.}338872$
i = $9\overset{\circ}{.}523308$
e = 0.086589
a = 3.498758

(909) Ulla (1971-1972)

t = 0h 6 Sep 1971
M = $73^\circ.825886$
ω = $226^\circ.996753$
Ω = $146^\circ.849739$
i = $18^\circ.811917$
e = 0.089385
a = 3.545228

(909) Ulla (1972-1973)

t = 0h 10 Oct 1972
M = $130^\circ.636997$
ω = $228^\circ.777156$
Ω = $146^\circ.921183$
i = $18^\circ.741364$
e = 0.091502
a = 3.546433

Since, in most cases, more than three positions of the minor planet were available, the next step was to use the remaining positions to improve the orbit. Another of the programs written by Watson (1974) was used for this; it uses the method devised by Porter (1949) for calculating corrections to the orbital elements. In each case, the elements of the preliminary orbit were fed in, together with all the available positions (including the three which were used in the preliminary calculation). The program calculated the residuals of the various positions from the original orbit (they were necessarily zero, or almost so, for the three positions already used); then, after calculating the corrections to the orbital elements, it re-calculated the residuals, this time from the "improved" orbit.

Unfortunately this program did not always produce the required results; for example, if any of the original residuals were too large, the "improved" orbit was liable to give considerably worse positions than the original one. In some cases, this could be prevented by removing those observations which gave the largest original residuals. In other cases, better results were obtained by varying the choice of three positions for the preliminary orbit determination. Various orbits were thus obtained for each planet, and that one chosen which gave the smallest residuals for all the available observations.

The final orbits determined for the planets are shown below, together with a list of the observations used in the determination, and a list of the residuals in all the available observations calculated from that orbit. For comparison, the corresponding elements from the "Ephemerides of Minor Planets" are given alongside in brackets; the mean anomaly M is extrapolated to the epoch of the new orbit, using the mean daily motion given in the "Ephemerides of Minor Planets". In the case of (909) Ulla, a new orbit was published in the "Ephemerides of Minor Planets" for 1977; both are shown here.

(87) Sylvia

e =	0.09446047	(0.099)
a =	3.48155544	(3.4812)
t =	0h 14 Nov 1973	(0h 2 Dec 1962)
M =	91° 41′ 26.″79	((98° 2′ 50.″4))
ω =	272 1 38.53	(265 33 28.8)
i =	10 54 28.59	(10 51 10.8)
Ω =	73 21 53.28	(74 4 44.4)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	$d\alpha$	dec.	$d\delta$
99	28/11/73	01^h 21^m 34^s	6^h 37^m 29.^{s}582	0.^{s}065	+27° 15′ 34.″00	0.″26
105	26/12/73	01 14 42	6 16 2.069	0.029	+28 39 57.48	0.36
115	17/ 1/74	19 40 46	5 57 50.547	0.031	+29 18 42.81	0.15

Residuals from All Observations

plate	$d\alpha$	$d\delta$
90	−2.″90	−0.″41
99	−0.01	−0.00
102	+2.26	−0.14
105	−0.01	−0.00
108	−1.70	−0.30
112	−0.98	+0.34
115	−0.01	−0.00

(107) Camilla

e =	0.07327751	(0.070)
a =	3.48882346	(3.4895)
t =	0h 14 Nov 1973	(0h 15 Jan 1947)
M =	345°22′55.″19	((333°40′43.″4))
ω =	291 58 30.32	(303 34 15.6)
i =	9 56 55.74	(9 55 19.2)
Ω =	174 3 18.70	(174 39 50.4)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	dα	dec.	dδ
98	28/11/73	00ʰ54ᵐ39ˢ	6ʰ48ᵐ54.ˢ400	0.ˢ033	+ 9°37′44.″04	0.″57
100	23/12/73	04 10 23	6 33 15.149	0.023	+ 9 18 36.33	0.20
114	17/ 1/74	19 14 00	6 14 28.535	0.029	+10 4 3.23	0.31

Residuals from All Observations

plate	dα	dδ
98	−0.″00	−0.″00
100	−0.00	−0.00
104	−0.27	−0.57
107	−1.37	−0.05
111	−0.62	−0.47
114	−0.00	−0.00

(334) Chicago (1971-1972)

e =	0.24428750	(0.057)
a =	4.36656909	(3.8854)
t =	0h 6 Sep 1971	(0h 15 Jan 1947)
M =	298°48′30.″53	((137°13′35.″4))
ω =	27 23 14.90	(163 43 1.0)
i =	4 7 19.59	(4 37 33.6)
Ω =	136 39 47.21	(131 31 26.4)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	$d\alpha$	dec.	$d\delta$
19	14/ 1/72	21ʰ41ᵐ28ˢ	5ʰ38ᵐ57.ˢ702	0.ˢ220	+19°34′37.″48	0.″61
23(a)	20/ 1/72	00 17 19	5 36 20.245	0.026	+19 38 52.34	0.78
26	20/ 1/72	23 07 54	5 35 53.624	0.024	+19 39 43.04	0.37

Residuals from All Observations

plate	$d\alpha$	$d\delta$
17	−0.″90	−4.″83
19	+0.01	+0.00
20	+3.75	−0.26
22	+0.36	−1.13
23(a)	+0.01	+0.00
23(b)	+3.05	+0.68
26	+0.01	+0.00

(334) Chicago (1972-1973)

e =	0.07908853	(0.057)
a =	3.81448678	(3.8854)
t =	0h 10 Oct 1972	(0h 15 Jan 1947)
M =	142° 2′ 14.″02	((291°39′17.″4))
ω =	204 48 36.76	(163 43 1.20)
i =	4 53 51.94	(4 37 33.6)
Ω =	131 10 1.46	(131 31 26.4)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	$d\alpha$	dec.	$d\delta$
43	7/ 2/73	$01^h20^m12^s$	$9^h17^m34.^s262$	$0.^s022$	+16°27′ 5.″78	0.″10
49	28/ 2/73	03 40 01	9 4 27.367	0.018	+17 40 35.29	0.26
53	8/ 3/73	02 47 37	9 0 32.701	0.041	+18 2 9.73	0.41

Residuals from All Observations

plate	$d\alpha$	$d\delta$
43	+0.″33	+0.″03
47	+0.68	+0.02
49	+0.00	-0.00
51	+0.18	-0.52
53	+0.00	-0.00

(414) Liriope

e = 0.08658933 (0.077)
a = 3.49881874 (3.5072)
t = 0h 14 Nov 1973 (0h 2 Dec 1962)
M = 357°10′31.″48 ((359°27′ 2.″8))
ω = 317 13 12.69 (313 44 45.6)
i = 9 31 28.68 (9 32 31.2)
Ω = 111 19 57.88 (111 53 24.0)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	$d\alpha$	dec.	$d\delta$
81	22/10/73	21^h 53^m 05^s	4^h 57^m 22.^s472	0.^s085	+12° 58′ 57.″59	0.″79
82	22/10/73	22 23 30	4 57 22.225	0.061	+12 58 56.70	0.52
83	26/10/73	23 00 37	4 56 12.178	0.048	+12 54 1.65	0.37
85	27/10/73	00 02 57	4 56 11.274	0.017	+12 53 59.74	0.38
86	29/10/73	02 06 44	4 55 26.465	0.042	+12 51 31.56	0.25
88	23/11/73	04 34 16	4 40 18.809	0.230	+12 30 59.73	1.39
97	28/11/73	00 25 44	4 36 31.690	0.030	+12 30 12.30	0.93
101	24/12/73	00 12 14	4 17 4.768	0.033	+12 52 56.30	0.19
110	15/ 1/74	19 45 38	4 8 7.956	0.058	+13 53 40.68	0.64

Residuals from All Observations

plate	$d\alpha$	$d\delta$
81	−0.″33	−0.″42
82	+0.67	+0.20
83	+0.12	−0.68
85	−0.04	+0.43
86	−0.14	+0.52
88	−0.25	+0.56
97	+0.24	−0.13
101	−0.06	−0.33
110	+0.00	−0.14

(909) Ulla (1971-1972)

e = 0.09068511	(0.091)	(0.093)
a = 3.54632563	(3.5453)	(3.5513)
t = 0h 6 Sep 1971	(0h 24 May 1968)	(0h 2 Dec 1962)
M = 73°17′ 9″.47	((71°37′54″.0))	((67°30′24″.4))
ω = 227 22 13.99	(229 1 51.6)	(232 6 7.2)
i = 18 48 6.10	(18 45 7.2)	(18 47 42.0)
Ω = 146 51 52.11	(146 58 4.8)	(147 11 49.2)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	dα	dec.	dδ
13	22/12/71	02ʰ03ᵐ02ˢ	8ʰ14ᵐ9ˢ.880	0ˢ.014	+ 6°13′47″.85	0″.21
24	20/ 1/72	02 38 56	7 55 31.486	0.034	+ 8 12 27.72	0.15
33(a)	12/ 2/72	21 8 13	7 40 3.416	0.015	+10 38 23.05	0.15

Residuals from All Observations

plate	dα	dδ
12	−0″.01	+1″.46
13	+0.56	+1.00
14	+0.86	+0.31
15	+0.64	−0.26
16	+0.12	−0.28
24	−0.06	−0.97
25	+2.19	−0.31
27	−0.09	−0.95
29	+0.33	−0.82
31	+0.38	−0.02
33(a)	−0.07	−0.38
33(b)	+0.98	+0.01
34	+1.14	+0.23
35	+0.19	+1.27

(909) Ulla (1972-1973)

e = 0.09161758	(0.091)	(0.093)
a = 3.54617087	(3.5453)	(3.5513)
t = 0h 10 Oct 1972	(0h 24 May 1968)	(0h 2 Dec 1962)
M = 130°55′31″.49	((130°41′22″.4))	((126°24′54″.0))
ω = 228 30 51.21	(229 1 51.6)	(232 6 7.2)
i = 18 45 1.34	(18 45 7.2)	(18 47 42.0)
Ω = 146 55 59.75	(146 58 4.8)	(147 11 49.2)

Observations Used to Determine Orbit

plate	date	U.T.	R.A.	dα	dec.	dδ
46	10/ 2/73	00ʰ42ᵐ28ˢ	12ʰ 4ᵐ15ˢ.173	0ˢ.086	+ 9°23′57″.25	0″.58
54	8/ 3/73	3 18 32	11 51 56.584	0.080	+12 39 27.61	0.35
61	5/ 4/73	02 23 18	11 35 18.389	0.020	+15 35 55.82	0.16

Residuals from All Observations

plate	dα	dδ
42	+4″.70	+2″.80
44	+1.59	+5.20
46	+0.01	−0.01
50	+1.20	−1.48
52	+29.56	+128.43
54	+0.01	−0.01
56	+0.48	−0.44
57	−0.10	−0.47
58	+1.55	−1.67
59	−0.66	−0.93
60	−0.59	+1.12
61	+0.01	−0.01
62	−0.24	−0.64

Let us now consider the changes that appear in these new orbits, in comparison with the "Ephemerides of Minor Planets" orbits.

(87) Sylvia

Over an interval of almost 11 years, the eccentricity of this orbit has decreased by about 4%, while the semi-major axis has increased by a much smaller proportion (0.01%). The perihelion has advanced by about 6.5 degrees. The inclination has decreased slightly (about 0.4%), and the ascending node has retrogressed very slightly (0.7 degrees). The actual mean longitude of the planet $M+\varpi$ according to the two orbits (using the extrapolated M from the earlier orbit) differs by only 0.1 degrees. The ratio of its mean motion to that of Jupiter remains at 1.83.

(107) Camilla

Over a time-interval more than twice that above - 27 years - the changes in this orbit are of very similar magnitude, though generally opposite direction. The eccentricity has increased by about 4%, and the semi-major axis decreased by 0.02%. The perihelion has retrogressed by 11.5 degrees (that is, it has moved at a rate very comparable with that of (87) Sylvia, though in the opposite direction). The inclination has increased by about 0.3%, and the ascending node has again retrogressed, by 0.6 degrees. The actual mean longitude of the planet again agrees to the nearest 0.1 degree, and the ratio of mean motion to that of Jupiter is again effectively unchanged at 1.82.

(334) Chicago

In this case, a wide variety of different orbits was obtained, for both seasons, by selecting different sets of positions for the preliminary orbit. (In no case was the second computer program able to fit an improved orbit to all the observations.) These orbits had eccentricities ranging from 0.08 to 0.56, and semi-major axes ranging from 3.8 to 4.7. The orientations of the perihelion were mostly in the region of 210 degrees, but in one case (that shown here for 1971-1972) it was shifted from this by about 180 degrees (that is, the perihelion and aphelion were exchanged). The inclinations and the longitudes of ascending node were less inconsistent, the inclination ranging from 4.0 to 5.5 degrees, and the longitude of ascending node being always about 130 degrees. The orbits chosen here for (334) Chicago are those which gave the smallest residuals for all the positions available, but little faith should be placed in them. It is clear that these sets of data are numerically unstable for the production of orbits. Consequently it is not surprising that the agreement between these two orbits and that given in the "Ephemerides of Minor Planets" should be so much worse than for the other planets. In the case of the first orbit, the actual mean longitude of the planet is shifted by over

25 degrees; in the second, by about 5 degrees. (In fact, there was on the contrary very close agreement between the positions predicted for the planet in the "Ephemerides of Minor Planets" and the positions at which it was actually observed.) Clearly no meaningful conclusions can be drawn from these orbits.

(414) Liriope

Over an interval of 11 years, some of the parameters of this planet's orbit have changed considerably: the eccentricity has increased by over 12%, and the semi-major axis decreased by about 0.2%. These changes both agree in direction with those of (107) Camilla, though they are much larger; however, in this case the perihelion has advanced, and by only 3.5 degrees. The inclination has decreased slightly (0.2%), and the ascending node, like those of both (87) Sylvia and (107) Camilla, has retrogressed by about 0.6 degrees. The mean longitude of the planet has changed by just over a degree, possibly confirming a significant change in the orbit (the observed positions did actually differ from those predicted in the "Ephemerides of Minor Planets" by about half a degree); but the ratio of mean motion to that of Jupiter again remains essentially unchanged at 1.81.

(909) Ulla

Here we have a total of four orbits to compare, two published in the "Ephemerides of Minor Planets", and two derived from observations. Each of the latter was calculated from three observations only, and both give comparably small average residuals for the reasonably large number of other observations available (13 and 14 observations respectively, giving average residuals of around 0.6 seconds in either coordinate). The exception is plate number 52, which persisted in giving large residuals for all calculated orbits, and which could not itself be used to calculate any orbit, without giving almost equally large residuals for all the other observations. The figures for this observation have been checked and re-checked, and the only possible explanation seems to be that the clock was mis-read when the plate was taken - an error now impossible to confirm or to eliminate.

Comparing all four orbits, we have a change in direction of movement of all the orbital parameters over the ten years covered. The eccentricity, decreasing from 1962 to 1971, increases a little again by 1972; the total change is just under 3%, the final increase being about 1%, of the average value. The semi-major axis decreases over the first interval (to 1968), and then rises a little by 1971 and stays almost constant the next year; here the total change is less than 0.2%. The perihelion retrogresses steadily, at about 0.5 degrees per year, until 1971, and then advances more than 1 degree by 1972. The inclination decreases a little over the first interval, then more rapidly over the second (a total change of about 1%), and finally rises back almost to its first level. The ascending node shows very similar behaviour, retrogressing slowly and then more rapidly over a total of 1.3 degrees, and finally re-advancing a little. The changes in actual mean longitude of the planet amount to 0.6 degrees over the first interval, 0.1 degrees over the second and 0.3 degrees over the third. The ratio of mean motion to that of Jupiter changes only from 1.77 to 1.78.

If these various changes are real - and there seems to be no obvious reason to doubt either of the new orbits more than the other - then it would appear that all the orbital parameters are fluctuating in such a way that they all pass through a minimum or a maximum at around the same time, which in this case would seem to be around 1971.

All the orbits discussed here have undergone changes, but none very large; they are all compatible with the hypothesis that the orbits are basically stable but that the parameters undergo small fluctuations. None of them has changed enough to cause the minor planet to cross the exact point of commensurability with Jupiter, so, according to E. W. Brown as quoted by Schweizer (1969), no fundamental changes in the orbits should be expected. In the case of (414) Liriope, some of the changes in the parameters appear to be much larger than those of other planets with similar orbits; this may indicate that these parameters are at the most-rapidly changing phase of their oscillations.

The natures and periods of these fluctuations, and the likelihood of their occurring with the same period and phase in different orbital elements (as apparently shown here by (909) Ulla) is discussed in the next section, where orbits are integrated numerically over long time-intervals.

III - Theoretical Investigations

The theoretical approach, based on Schubart (1968), was to determine the differential equations for the changes in the orbital parameters of the minor planet, due to the perturbations of Jupiter, in such a way that short- and long-period effects could be separated. The short-period effects were then averaged, by allowing only the longitudes of the two planets to vary around a complete period of the commensurability while keeping the other parameters constant. At each point the derivatives of the orbital elements were calculated, and at the end of the loop they were averaged; these average values were then applied to all the parameters as one step in a numerical integration process.

The ratio of the commensurability was taken to be $(p+q)/p$, where p and q are small integers. In the case of (153) Hilda and other planets at the 3/2 commensurability, we have $p = 2$ and $q = 1$. For most of the planets treated in this investigation, the ratio of mean motions was approximately 9/5, and consequently $p = 5$, $q = 4$.

The equations used are adapted from those first derived by Gauss for the derivatives of the common orbital elements a, e, i, Ω, ϖ and ε, in terms of mutually perpendicular components of the disturbing force; see, for example, Brouwer and Clemence (1961).

The notation here is slightly different from that of Brouwer and Clemence. The components of the disturbing force are taken as: S, acting outwards along the radius vector of the minor planet; T, acting perpendicularly to the radius vector in the plane of the orbit; and W, acting perpendicularly to the plane of the orbit; the whole (S,T,W) forming a right-handed set.

Throughout this thesis, the true anomaly is designated by v, the eccentric anomaly by E, and the inclination by i. Other notation is the same as that of Brouwer and Clemence.

With these changes, their equations (page 301) become:

$$\frac{da}{dt} = \frac{2}{n\sqrt{(1-e^2)}}\left(S\ e \sin(v) + T\ \frac{p}{r}\right)$$

$$\frac{de}{dt} = \frac{\sqrt{(1-e^2)}}{n\ a}\left(S \sin(v) + T\ (\cos(E) + \cos(v))\right)$$

$$\frac{di}{dt} = \frac{1}{n\ a\sqrt{(1-e^2)}}\ W\ \frac{r}{a}\cos(\omega+v)$$

$$\sin(i)\ \frac{d\Omega}{dt} = \frac{1}{n\ a\sqrt{(1-e^2)}}\ W\ \frac{r}{a}\sin(\omega+v) \tag{1}$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{(1-e^2)}}{n\ a\ e}\left(-S\cos(v) + T\left(\frac{r}{p}+1\right)\sin(v)\right) + 2\sin^2\left(\frac{i}{2}\right)\frac{d\Omega}{dt}$$

$$\frac{d\varepsilon}{dt} = -\frac{2\ r}{na^2}\ S + \frac{e^2}{1+\sqrt{(1-e^2)}}\ \frac{d\varpi}{dt} + 2\sqrt{(1-e^2)}\ \sin^2\left(\frac{i}{2}\right)\frac{d\Omega}{dt}$$

In our integration, the orbital elements (a, e, i, Ω, ϖ, ε) are replaced by the set (G, μ, ψ_1, ψ_2, χ_1, χ_2), which are combinations of the original set, chosen to make the calculations simpler. They are:

$$G = \sqrt{a(1-e^2)}$$

$$\mu = l - \frac{p+q}{p} l_J$$

$$\psi_1 = e \cos(\varpi)$$

$$\psi_2 = e \sin(\varpi)$$

$$\chi_1 = \sin(i) \cos(\Omega)$$

$$\chi_2 = \sin(i) \sin(\Omega)$$

where l is the mean longitude of the minor planet (not the mean anomaly, as in the notation of Brouwer and Clemence). It is measured from the longitude of the perihelion of Jupiter, which is taken as the origin of longitudes (so that $\varpi_J = 0$). Similarly l_J is the mean longitude of Jupiter; and $(p+q)/p$ is the ratio of the mean motions of the minor planet and Jupiter. This means that μ is a constant if the commensurability is perfect at all times; in practice, it is a slowly-changing variable. This use of μ eliminates the need for using rapidly-changing variables such as l or ε. G is not to be confused with the gravitational constant $\mathcal{G} = 1$ in our units.

The first four of these variables are those used by Schubart (1968); the remaining two are introduced here to complete the set. The transforming of e and ϖ into $\psi_1 = e \cos(\varpi)$ and $\psi_2 = e \sin(\varpi)$ is a standard technique for avoiding the problems that would arise if e should at any time become zero, and ϖ consequently indeterminate. A similar argument applies to the transformation of i and Ω into $\chi_1 = \sin(i)\cos(\Omega)$ and $\chi_2 = \sin(i)\sin(\Omega)$; the use of tan(i) instead of sin(i) would have had the same effect. Both transformations may be found in Roy (1978); Schubart in his recent paper (1978) uses tan(i/2).

Before using the equations (1) to calculate the derivatives of the elements (G, μ, ψ_1, ψ_2, χ_1, χ_2), some simplifications are possible.

The unit of length is chosen to be the semi-major axis of Jupiter's orbit; i.e. $a_J = 1$. The unit of mass is chosen to be the mass of the sun; i.e. $m_\odot = 1$. The unit of time is then chosen to be such that the gravitational constant $\mathcal{G}$ should be exactly 1. This gives the period of Jupiter as $0.999523 * 2\pi$ time-units. Since the period of Jupiter is about 11.86 years, this means that a time-unit is approximately 1.89 years.

As this problem is being treated as a version of the restricted three-body problem, (see for example Brouwer and Clemence (1961)), the mass m of the minor planet is neglected. So we have

$$n^2 a^3 = \mathcal{G}(m + m_\odot) = 1$$

and consequently we can replace n by $a^{-1.5}$ throughout. Also the parameter $p = a(1-e^2)$ can be replaced by G^2.

So the equations (1) become:

$$\frac{da}{dt} = \frac{2a^2}{G} (S\ e \sin(v) + T(1 + e \cos(v)))$$

$$\frac{de}{dt} = G(S \sin(v) + T(\cos(E) + \cos(v)))$$

$$\frac{di}{dt} = W \frac{r}{G} \cos(\omega+v)$$

$$\frac{d\Omega}{dt} = W \frac{r}{G} \sin(\omega+v) \operatorname{cosec}(i)$$

$$\frac{d\bar{\omega}}{dt} = - G\ S \frac{\cos(v)}{e} + G\ T\left(\frac{r}{G^2} + 1\right) \frac{\sin(v)}{e}$$

$$+ W \frac{r}{G} \sin(\omega+v) \tan(0.5\ i)$$

$$\frac{d\varepsilon}{dt} = - G\ S\left(2 \frac{r}{G^2} \sqrt{(1-e^2)} + \frac{e \cos(v)}{1 + \sqrt{(1-e^2)}}\right)$$

$$+ G\ T\left(\frac{r}{G^2} + 1\right) \frac{e \sin(v)}{1 + \sqrt{(1-e^2)}}$$

$$+ W \frac{r}{G} \sin(\omega+v) \tan(0.5\ i)$$

We can now use these expressions to calculate the derivatives of the elements (G, μ, ψ_1, ψ_2, χ_1, χ_2). We begin with G.

$$G = \sqrt{a(1-e^2)}$$

so $$\frac{dG}{dt} = \frac{1}{2G}\left((1-e^2)\frac{da}{dt} - 2ae\frac{de}{dt}\right)$$

$$= \frac{1}{2G}\left((1-e^2)\frac{2a^2}{G}(S\ e\ \sin(v) + T(1 + e\cos(v)))\right.$$

$$\left. - 2aeG(S\ \sin(v) + T(\cos(E) + \cos(v)))\right)$$

$$= a(S\ e\ \sin(v) + T + T\ e\cos(v))$$

$$- ae(S\ \sin(v) + T\cos(E) + T\cos(v))$$

$$= a\ T(1 - e\cos(E))$$

$$= r\ T$$

so $$\frac{dG}{dt} = r\ T$$

****

$$\mu = l - \frac{p+q}{p} l_{\mathcal{J}} = nt + \varepsilon - \frac{p+q}{p} l_{\mathcal{J}}$$

$$\text{so } \frac{d\mu}{dt} = n + \frac{d\varepsilon}{dt} - \frac{p+q}{p}\frac{dl_{\mathcal{J}}}{dt}$$

$$= a^{-1.5} - \frac{p+q}{p} n_{\mathcal{J}} + \frac{d\varepsilon}{dt}$$

$$\text{so } \frac{d\mu}{dt} = a^{-1.5} - \frac{p+q}{p} n_{\mathcal{J}} - G\ S(2\ \frac{r}{G^2}\sqrt{(1-e^2)} + \frac{e\ \cos(v)}{1 + \sqrt{(1-e^2)}})$$

$$+ G\ T(\frac{r}{G^2} + 1)\frac{e\ \sin(v)}{1 + \sqrt{(1-e^2)}} + W\ \frac{r}{G} \sin(\omega+v)\ \tan(0.5\ i)$$

****

$$\psi_1 = e \cos(\bar{\omega})$$

so $$\frac{d\psi_1}{dt} = \cos(\bar{\omega}) \frac{de}{dt} - e \sin(\bar{\omega}) \frac{d\bar{\omega}}{dt}$$

$$= \cos(\bar{\omega})\, G\, (S \sin(v) + T(\cos(E) + \cos(v)))$$

$$- e \sin(\bar{\omega}) (- G\, S \frac{\cos(v)}{e} + G\, T(\frac{r}{G^2} + 1) \frac{\sin(v)}{e}$$

$$+ W \frac{r}{G} \sin(\omega + v) \tan(0.5\, i)$$

For cos(E) we may write

$$\cos(E) = \frac{\cos(v) + e}{1 + e \cos(v)}$$

$$= \frac{r}{G^2} (\cos(v) + e)$$

$$\text{so } \frac{d\psi_1}{dt} = G\ S\ (\cos(\varpi)\sin(v) + \sin(\varpi)\ \cos(v))$$

$$+ G\ T(\cos(\varpi)\ \frac{r}{G^2}\ (\cos(v) + e) + \cos(\varpi)\ \cos(v)$$

$$- (\frac{r}{G^2} + 1)\ \sin(\varpi)\ \sin(v))$$

$$- W\ \frac{r}{G}\ \sin(\omega+v)\ \tan(0.5\ i)\ e\ \sin(\varpi)$$

$$= G\ S\ \sin(v+\varpi)$$

$$+ G\ T(\cos(v+\varpi)(1 + \frac{r}{G^2}) + \frac{r}{G^2}\ e\ \cos(\varpi))$$

$$- W\ \frac{r}{G}\ \sin(\omega+v)\ \tan(0.5\ i)\ e\ \sin(\varpi)$$

$$\text{so } \frac{d\psi_1}{dt} = G\ S\ \sin(v+\varpi)$$

$$+ G\ T(\cos(v+\varpi)(1+ \frac{r}{G^2}) + \frac{r}{G^2}\ \psi_1)$$

$$- W\ \frac{r}{G}\ \sin(\omega+v)\ \tan(0.5\ i)\psi_2$$

****

$$\psi_2 = e \sin(\varpi)$$

so $$\frac{d\psi_2}{dt} = \sin(\varpi) \frac{de}{dt} + e \cos(\varpi) \frac{d\varpi}{dt}$$

$$= \sin(\varpi)\ G(S \sin(v) + T(\cos(E) + \cos(v)))$$

$$+ e \cos(\varpi)(-G\ S \frac{\cos(v)}{e} + G\ T(\frac{r}{G^2} + 1)\frac{\sin(v)}{e}$$

$$+ W \frac{r}{G} \sin(\omega+v) \tan(0.5\ i)\ )$$

Again writing $\cos(E) = \frac{r}{G^2} (\cos(v) + e)$, we have

$$\frac{d\psi_2}{dt} = G\ S(\sin(\varpi)\sin(v) - \cos(\varpi)\cos(v))$$

$$+ G\ T(\sin(\varpi) \frac{r}{G^2} (\cos(v) + e) + \sin(\varpi)\cos(v)$$

$$+ (\frac{r}{G^2} + 1) \cos(\varpi)\sin(v))$$

$$+ W \frac{r}{G} \sin(\omega+v) \tan(0.5\ i)\ e \cos(\varpi)$$

$$= - G\ S \cos(v+\varpi)$$

$$+ G\ T(\sin(v+\varpi)(1+ \frac{r}{G^2}) + \frac{r}{G^2}\ e \sin(\varpi))$$

$$+ W \frac{r}{G} \sin(\omega+v) \tan(0.5\ i)\ e \cos(\varpi)$$

so $$\frac{d\psi_2}{dt} = -G\,S\,\cos(v+\varpi)$$

$$+ G\,T\left(\sin(v+\varpi)\left(1+\frac{r}{G^2}\right)+\frac{r}{G^2}\psi_2\right)$$

$$+ W\,\frac{r}{G}\,\sin(\omega+v)\,\tan(0.5\,i)\psi_1$$

****

$$\chi_1 = \sin(i)\ \cos(\Omega)$$

$$\text{so } \frac{d\chi_1}{dt} = \cos(i)\ \cos(\Omega)\ \frac{di}{dt} - \sin(i)\ \sin(\Omega)\ \frac{d\Omega}{dt}$$

$$= \cos(i)\ \cos(\Omega)\ W\ \frac{r}{G}\ \cos(\omega+v)$$

$$- \sin(i)\ \sin(\Omega)\ W\ \frac{r}{G}\ \sin(\omega+v)\ \mathrm{cosec}(i)$$

$$\text{so } \frac{d\chi_1}{dt} = W\ \frac{r}{G}\ (\cos(i)\ \cos(\Omega)\ \cos(\omega+v) - \sin(\Omega)\ \sin(\omega+v))$$

****

$$\chi_2 = \sin(i)\ \sin(\Omega)$$

$$\text{so } \frac{d\chi_2}{dt} = \cos(i)\ \sin(\Omega)\ \frac{di}{dt} + \sin(i)\ \cos(\Omega)\ \frac{d\Omega}{dt}$$

$$= \cos(i)\ \sin(\Omega)\ W\ \frac{r}{G}\ \cos(\omega+v)$$

$$+ \sin(i)\ \cos(\Omega)\ W\ \frac{r}{G}\ \sin(\omega+v)\ \mathrm{cosec}(i)$$

$$\text{so } \frac{d\chi_2}{dt} = W\ \frac{r}{G}\ (\cos(i)\ \sin(\Omega)\ \cos(\omega+v) + \cos(\Omega)\ \sin(\omega+v))$$

****

It will be noticed that, in these six equations for the derivatives of the six orbital elements, the components S and T generally appear multiplied by G. Following Schubart (1968), we therefore define new components $S_{\sigma} = G\,S$ and $T_{\sigma} = G\,T$. For the sake of consistency, a similar component $W_{\sigma} = G\,W$ is defined, with the incidental advantage that the term r/G can be replaced by r/G^2, which occurs frequently elsewhere.

We replace the angle $v+\varpi$ by λ, which is the "true longitude" of the planet, measured from the origin of longitudes; and we rewrite $\omega+v$ $(=\varpi-\Omega+v)$ as u, sometimes called the "argument of the latitude".

These alterations give us the six equations actually used in the integration program:

$$\frac{dG}{dt} = \frac{r}{G} T_J$$

$$\frac{d\mu}{dt} = a^{-1.5} - \frac{p+q}{p} n_J - S_J \left(2 \frac{r}{G^2} \sqrt{(1-e^2)} + \frac{e \cos(v)}{1+\sqrt{(1-e^2)}}\right)$$

$$+ T_J \left(\frac{r}{G^2} + 1\right) \frac{e \sin(v)}{1+\sqrt{(1-e^2)}} + W_J \frac{r}{G^2} \sin(u) \tan(0.5\ i)$$

$$\frac{d\psi_1}{dt} = S_J \sin(\lambda) + T_J \cos(\lambda) \left(\frac{r}{G^2} + 1) + \frac{r}{G^2}\psi_1\right)$$

$$- W_J \frac{r}{G^2} \sin(u) \tan(0.5\ i)\psi_2$$

$$\frac{d\psi_2}{dt} = - S_J \cos(\lambda) + T_J (\sin(\lambda) \left(\frac{r}{G^2} + 1\right) + \frac{r}{G^2} \psi_2)$$

$$+ W_J \frac{r}{G^2} \sin(u) \tan(0.5\ i)\psi_1$$

$$\frac{d\chi_1}{dt} = W_J \frac{r}{G^2} (\cos(i) \cos(\Omega) \cos(u) - \sin(\Omega) \sin(u))$$

$$\frac{d\chi_2}{dt} = W_J \frac{r}{G^2} (\cos(i) \sin(\Omega) \cos(u) + \cos(\Omega) \sin(u))$$

These equations confirm Schubart's remark (1978) that the formulae for the derivatives of the last two parameters (those that depend on the inclination) are simple by comparison with those for the first four.

The next stage is to determine S_J, T_J and W_J, by determining the components S, T and W of the disturbing force. We consider the usual expression for the disturbing function for the restricted three-body problem:

$$R = \mathcal{G} m_J \left(\frac{1}{\Delta} - \frac{\underline{r} \cdot \underline{r}_J}{r_J^3} \right)$$ (See, for example, Roy (1978).)

where: $\mathcal{G}$ is the gravitational constant, which is equal to 1 in our units; $\underline{r}$ and $\underline{r}_J$ are the radius vectors of the minor planet and of Jupiter; and Δ is the distance between the minor planet and Jupiter. Then, in any rectangular coordinate system (x,y,z), the components of the disturbing force $\underline{F}$ are:

$$F_x = \frac{\partial R}{\partial x} \qquad F_y = \frac{\partial R}{\partial y} \qquad F_z = \frac{\partial R}{\partial z}$$

Now $\underline{r} \cdot \underline{r}_J = x\, x_J + y\, y_J + z\, z_J$

and $\Delta^2 = (x-x_J)^2 + (y-y_J)^2 + (z-z_J)^2$

$$\text{So } \frac{\partial R}{\partial x} = m_J \left(- \frac{x-x_J}{\Delta^3} - \frac{x_J}{r_J^3} \right)$$

$$= m_J \left(x_J \left(\frac{1}{\Delta^3} - \frac{1}{r_J^3} \right) - \frac{x}{\Delta^3} \right)$$

and similarly for $\frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial z}$.

So we have:

$$F_x = m_J(\ x_J(\Delta^{-3} - r_J^{-3}) - x\Delta^{-3}\ )$$
$$F_y = m_J(\ y_J(\Delta^{-3} - r_J^{-3}) - y\Delta^{-3}\ ) \qquad (2)$$
$$F_z = m_J(\ z_J(\Delta^{-3} - r_J^{-3}) - z\Delta^{-3}\ )$$

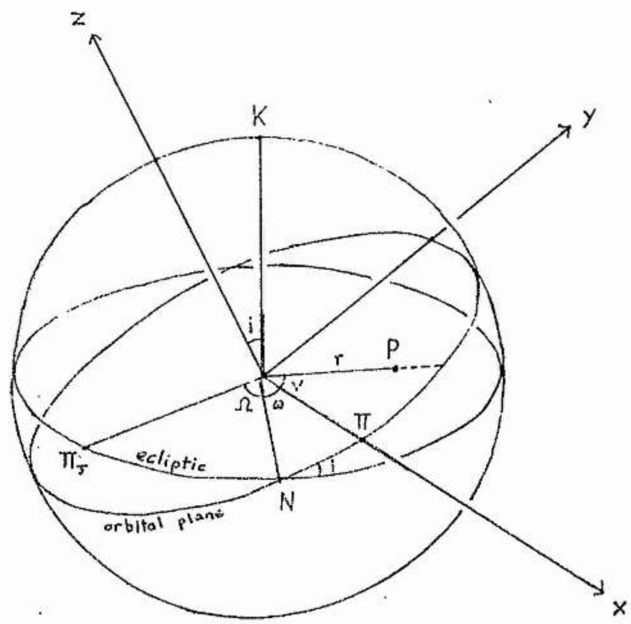
These expressions are true in any coordinate system (x,y,z), but it is simplest to choose a system with the x,y-plane in the ecliptic, and the x-axis through the origin of longitudes (i.e. the longitude of the perihelion of Jupiter). It is then necessary to find the coordinates (x,y,z) and (x_J,y_J,z_J) of the minor planet and of Jupiter, in this system.

For each planet the procedure is the same. We begin with a set of orbital coordinates (X,Y,Z), where the X,Y-plane is the plane of the planet's orbit, and the X-axis passes through the perihelion. This gives:

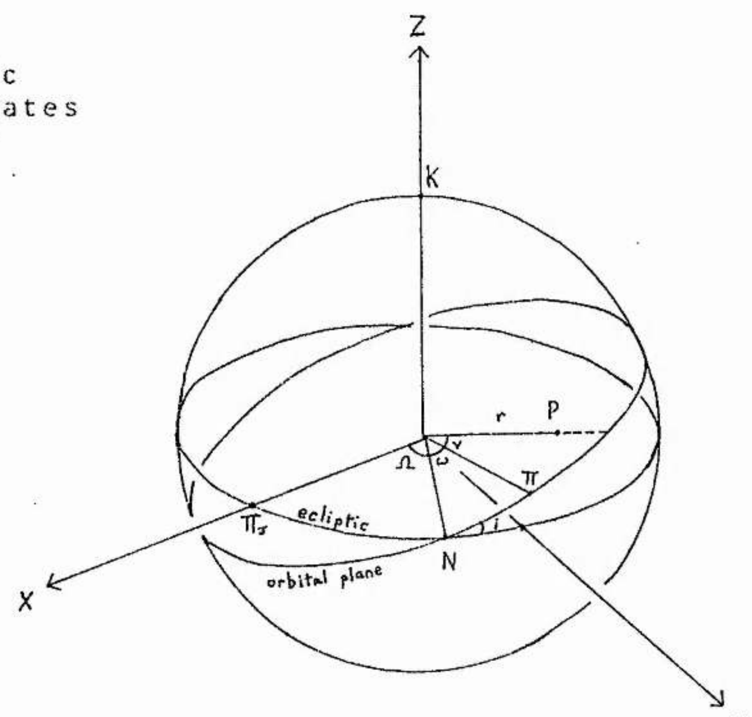
$$X = r\cos(v) \qquad Y = r\sin(v) \qquad Z = 0$$

To transform these orbital coordinates (X,Y,Z) into the ecliptic coordinates (x,y,z), three rotations are needed (see diagrams 7):

orbital
coordinates
(x, y, z)



ecliptic
coordinates
(X, Y, Z)



K is pole of ecliptic
N is rising node of orbit of planet on ecliptic
Π_J is perihelion of Jupiter (origin of longitudes)
π is perihelion of planet
P is position of planet

RELATIONSHIP BETWEEN ORBITAL and ECLIPTIC COORDINATES

Diagram 7

(a) rotation by $-\omega$ about the Z-axis (to move the X-axis to the ascending node)

(b) rotation by $-i$ about the new x-axis (to move the x,y-plane into the ecliptic)

(c) rotation by $-\Omega$ about the new z-axis (to move the x-axis to the origin of longitudes).

These can be expressed as a product of three rotation matrices:

$$\begin{bmatrix} \cos(\;) & \sin(\;) & 0 \\ -\sin(\;) & \cos(\;) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & \sin(i) \\ 0 & -\sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} \cos(\;) & \sin(\;) & 0 \\ -\sin(\;) & \cos(\;) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the orbital coordinates $\begin{bmatrix} r\cos(v) & r\sin(v) & 0 \end{bmatrix}$ by this product of matrices gives the ecliptic coordinates of the planet:

$$x = r\cos(v)\,(\cos(\omega)\cos(\Omega) - \sin(\omega)\sin(\Omega)\cos(i)) + r\sin(v)\,(-\sin(\omega)\cos(\Omega) - \cos(\omega)\sin(\Omega)\cos(i))$$

$$y = r\cos(v)\,(\cos(\omega)\sin(\Omega) + \sin(\omega)\cos(\Omega)\cos(i)) + r\sin(v)\,(-\sin(\omega)\sin(\Omega) + \cos(\omega)\cos(\Omega)\cos(i))$$

$$z = r\,(\cos(v)\sin(\omega)\sin(i) + \sin(v)\cos(\omega)\sin(i))$$

Rearranging, and writing $v+\omega = u$ as before, this gives:

$$x = r(\cos(u)\cos(\Omega) - \sin(u)\sin(\Omega)\cos(i))$$

$$y = r(\cos(u)\sin(\Omega) + \sin(u)\cos(\Omega)\cos(i))$$

$$z = r\sin(u)\sin(i)$$

These transformations apply to any planet. In the case of the minor planet, the equations above give the ecliptic coordinates of the minor planet directly. The coordinates for Jupiter are obtained by substituting the appropriate variables for Jupiter's orbit. Thus we have:

$$x_J = r_J(\cos(u_J)\cos(\Omega_J) - \sin(u_J)\sin(\Omega_J)\cos(i_J))$$
$$y_J = r_J(\cos(u_J)\sin(\Omega_J) + \sin(u_J)\cos(\Omega_J)\cos(i_J))$$
$$z_J = r_J\sin(u_J)\sin(i_J)$$

We can now obtain the ecliptic components of the disturbing force, by substituting these expressions for (x,y,z) and (x_J,y_J,z_J) into the equations (2). This gives:

$$F_x = m_J(r_J(\cos(u_J)\cos(\Omega_J) - \sin(u_J)\sin(\Omega_J)\cos(i_J))(\Delta^{-3}-r_J^{-3}) - (\cos(u)\cos(\Omega) - \sin(u)\sin(\Omega)\cos(i))\, r\,\Delta^{-3})$$

$$F_y = m_J(r_J(\cos(u_J)\sin(\Omega_J) + \sin(u_J)\cos(\Omega_J)\cos(i_J))(\Delta^{-3}-r_J^{-3}) - (\cos(u)\sin(\Omega) + \sin(u)\cos(\Omega)\cos(i))\, r\Delta^{-3})$$

$$F_z = m_J(r_J\sin(u_J)\sin(i_J)(\Delta^{-3} - r_J^{-3}) - \sin(u)\sin(i)\, r\,\Delta^{-3})$$

The final step is to transform these components (F_x, F_y, F_z), which are in the ecliptic, into the components (S,T,W) which are in the orbital plane of the minor planet. Once again, three rotations are necessary (see diagrams 7):

(a) rotation by Ω about the z-axis (to move the x-axis to the ascending node)

(b) rotation by i about the new x-axis (to move the x,y-plane into the orbital plane)

(c) rotation by u about the new z-axis (to move the x-axis to the longitude of the planet).

These can be expressed as the product of the three rotation matrices:

$$\begin{bmatrix} \cos(\ ) & -\sin(\ ) & 0 \\ \sin(\ ) & \cos(\ ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the components $\begin{bmatrix} F_x & F_y & F_z \end{bmatrix}$ by this product of matrices gives the required components:

$$\begin{aligned} S = {} & F_x (\cos(\Omega) \cos(u) - \sin(\Omega) \sin(u) \cos(i)) \\ & + F_y (\sin(\Omega) \cos(u) + \cos(\Omega) \sin(u) \cos(i)) \\ & + F_z \sin(u) \sin(i) \end{aligned}$$

$$\begin{aligned} T = {} & F_x (-\cos(\Omega) \sin(u) - \sin(\Omega) \cos(u) \cos(i)) \\ & + F_y (-\sin(\Omega) \sin(u) + \cos(\Omega) \cos(u) \cos(i)) \\ & + F_z \cos(u) \sin(i) \end{aligned}$$

$$W = F_x \sin(\Omega) \sin(i) - F \cos(\Omega) \sin(i) + F \cos(i)$$

We shall evaluate these individually.

$$S = (\cos(\Omega)\cos(u) - \sin(\Omega)\sin(u)\cos(i))$$
$$m_J(r_J(\cos(u_J)\cos(\Omega_J) - \sin(u_J)\sin(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3})$$
$$- (\cos(u)\cos(\Omega) - \sin(u)\sin(\Omega)\cos(i))\ r\,\Delta^{-3})$$
$$+ (\sin(\Omega)\cos(u) + \cos(\Omega)\sin(u)\cos(i))$$
$$m_J(r_J(\cos(u_J)\sin(\Omega_J) + \sin(u_J)\cos(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3})$$
$$- (\cos(u)\sin(\Omega) + \sin(u)\cos(\Omega)\cos(i))\ r\,\Delta^{-3})$$
$$+ \sin(u)\sin(i)\ m_J(r_J\sin(u_J)\sin(i_J)(\Delta^{-3} - r_J^{-3})$$
$$- \sin(u)\sin(i)\ r\,\Delta^{-3})$$

The coefficient of $m_J r_J(\Delta^{-3} - r_J^{-3})$ in this expression is:

$$(\cos(\Omega)\ \cos(u) - \sin(\Omega)\ \sin(u)\ \cos(i))$$
$$(\cos(u_J)\ \cos(\Omega_J) - \sin(u_J)\ \sin(\Omega_J)\ \cos(i_J))$$
$$+ (\sin(\Omega)\ \cos(u) + \cos(\Omega)\ \sin(u)\ \cos(i))$$
$$(\cos(u_J)\ \sin(\Omega_J) + \sin(u_J)\ \cos(\Omega_J)\ \cos(i_J))$$
$$+ \sin(u)\ \sin(i)\ \sin(u_J)\ \sin(i_J)$$

$$= \cos(u)\ \cos(u_J)\ \cos(\Omega)\ \cos(\Omega_J)$$
$$- \sin(u)\ \cos(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i)$$
$$- \cos(u)\ \sin(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i_J)$$
$$+ \sin(u)\ \sin(u_J)\ \sin(\Omega)\ \sin(\Omega_J)\ \cos(i)\ \cos(i_J)$$
$$+ \cos(u)\ \cos(u_J)\ \sin(\Omega)\ \sin(\Omega_J)$$
$$+ \sin(u)\ \cos(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i)$$
$$+ \cos(u)\ \sin(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i_J)$$
$$+ \sin(u)\ \sin(u_J)\ \cos(\Omega)\ \cos(\Omega_J)\ \cos(i)\ \cos(i_J)$$
$$+ \sin(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)$$

$$= \cos(u)\ \cos(u_J)\ \cos(\Omega-\Omega_J)$$
$$- \sin(u)\ \cos(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i)$$
$$+ \cos(u)\ \sin(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i_J)$$
$$+ \sin(u)\ \sin(u_J)\ \cos(\Omega-\Omega_J)\ \cos(i)\ \cos(i_J)$$
$$+ \sin(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)$$

The coefficient of $- m_J\ r\ \Delta^{-3}$ is

$$(\cos(\Omega)\ \cos(u) - \sin(\Omega)\ \sin(u)\ \cos(i))$$
$$(\cos(u)\ \cos(\Omega) - \sin(u)\ \sin(\Omega)\ \cos(i))$$
$$+ (\sin(\Omega)\ \cos(u) + \cos(\Omega)\ \sin(u)\ \cos(i))$$
$$(\cos(u)\ \sin(\Omega) + \sin(u)\ \cos(\Omega)\ \cos(i))$$
$$+ \sin(u)\ \sin(i)\ \sin(u)\ \sin(i)$$

$$= \cos^2(u)\ \cos^2(\Omega)$$
$$- 2\ \sin(u)\ \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i)$$
$$+ \sin^2(u)\ \sin^2(\Omega)\ \cos^2(i)$$
$$+ \cos^2(u)\ \sin^2(\Omega)$$
$$+ 2\ \sin(u)\ \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i)$$
$$+ \sin^2(u)\ \cos^2(\Omega)\ \cos^2(i)$$
$$+ \sin^2(u)\ \sin^2(i)$$

$$= 1$$

So the expression for S becomes:

$$S = m_{\mathfrak{J}}(r_{\mathfrak{J}}(\cos(u)\ \cos(u_{\mathfrak{J}})\ \cos(\Omega-\Omega_{\mathfrak{J}})$$
$$- \sin(u)\ \cos(u_{\mathfrak{J}})\ \sin(\Omega-\Omega_{\mathfrak{J}})\ \cos(i)$$
$$+ \cos(u)\ \sin(u_{\mathfrak{J}})\ \sin(\Omega-\Omega_{\mathfrak{J}})\ \cos(i_{\mathfrak{J}})$$
$$+ \sin(u)\ \sin(u_{\mathfrak{J}})\ \cos(\Omega-\Omega_{\mathfrak{J}})\ \cos(i)\ \cos(i_{\mathfrak{J}})$$
$$+ \sin(u)\ \sin(u_{\mathfrak{J}})\ \sin(i)\ \sin(i_{\mathfrak{J}}))\ (\Delta^{-3} - r_{\mathfrak{J}}^{-3})$$
$$- r\ \Delta^{-3})$$

$$
\begin{aligned}
T = {} & (-\cos(\Omega)\sin(u) - \sin(\Omega)\cos(u)\cos(i)) \\
& \quad m_J(r_J(\cos(u_J)\cos(\Omega_J) - \sin(u_J)\sin(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3}) \\
& \qquad - (\cos(u)\cos(\Omega) - \sin(u)\sin(\Omega)\cos(i))\ r\ \Delta^{-3}) \\
& + (-\sin(\Omega)\sin(u) + \cos(\Omega)\cos(u)\cos(i)) \\
& \quad m_J(r_J(\cos(u_J)\sin(\Omega_J) + \sin(u_J)\cos(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3}) \\
& \qquad - (\cos(u)\sin(\Omega) + \sin(u)\cos(\Omega)\cos(i))\ r\ \Delta^{-3}) \\
& \qquad + \cos(u)\sin(i)\ m_J(r_J\sin(u_J)\sin(i_J)\ (\Delta^{-3} - r_J^{-3}) \\
& \qquad\qquad - \sin(u)\sin(i)\ r\ \Delta^{-3})
\end{aligned}
$$

The coefficient of $m_J r_J(\Delta^{-3} - r_J^{-3})$ in this expression is:

$$
\begin{aligned}
& (-\cos(\Omega)\ \sin(u) - \sin(\Omega)\ \cos(u)\ \cos(i)) \\
& \quad (\cos(u_J)\ \cos(\Omega_J) - \sin(u_J)\ \sin(\Omega_J)\ \cos(i_J)) \\
& + (-\sin(\Omega)\ \sin(u) + \cos(\Omega)\ \cos(u)\ \cos(i)) \\
& \quad (\cos(u_J)\ \sin(\Omega_J) + \sin(u_J)\ \cos(\Omega_J)\ \cos(i_J)) \\
& + \cos(u)\ \sin(i)\ \sin(u_J)\ \sin(i_J)
\end{aligned}
$$

$$
\begin{aligned}
= {} & -\sin(u)\ \cos(u_J)\ \cos(\Omega)\ \cos(\Omega_J) \\
& - \cos(u)\ \cos(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i) \\
& + \sin(u)\ \sin(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i_J) \\
& + \cos(u)\ \sin(u_J)\ \sin(\Omega)\ \sin(\Omega_J)\ \cos(i)\ \cos(i_J) \\
& - \sin(u)\ \cos(u_J)\ \sin(\Omega)\ \sin(\Omega_J) \\
& + \cos(u)\ \cos(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i) \\
& - \sin(u)\ \sin(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i_J) \\
& + \cos(u)\ \sin(u_J)\ \cos(\Omega)\ \cos(\Omega_J)\ \cos(i)\ \cos(i_J) \\
& + \cos(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)
\end{aligned}
$$

$$
\begin{aligned}
= &- \sin(u)\ \cos(u_J)\ \cos(\Omega-\Omega_J) \\
&- \cos(u)\ \cos(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i) \\
&- \sin(u)\ \sin(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i_J) \\
&+ \cos(u)\ \sin(u_J)\ \cos(\Omega-\Omega_J)\ \cos(i)\ \cos(i_J) \\
&+ \cos(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)
\end{aligned}
$$

The coefficient of $-m_J\, r\, \Delta^{-3}$ is:

$$
\begin{aligned}
&(-\cos(\Omega)\ \sin(u) - \sin(\Omega)\ \cos(u)\ \cos(i)) \\
&\quad (\cos(u)\ \cos(\Omega) - \sin(u)\ \sin(\Omega)\ \cos(i)) \\
&+ (-\sin(\Omega)\ \sin(u) + \cos(\Omega)\ \cos(u)\ \cos(i)) \\
&\quad (\cos(u)\ \sin(\Omega) + \sin(u)\ \cos(\Omega)\ \cos(i)) \\
&+ \cos(u)\ \sin(i)\ \sin(u)\ \sin(i)
\end{aligned}
$$

$$
\begin{aligned}
= &- \sin(u)\ \cos(u)\ \cos^2(\Omega) \\
&- \cos^2(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&+ \sin^2(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&+ \sin(u)\ \cos(u)\ \sin^2(\Omega)\ \cos^2(i) \\
&- \sin(u)\ \cos(u)\ \sin^2(\Omega) \\
&+ \cos^2(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&- \sin^2(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&+ \sin(u)\ \cos(u)\ \cos^2(\Omega)\ \cos^2(i) \\
&+ \sin(u)\ \cos(u)\ \sin^2(i)
\end{aligned}
$$

$$= 0$$

So the expression for T becomes:

$$T = -m_J r_J (\sin(u) \cos(u_J) \cos(\Omega-\Omega_J)$$
$$+ \cos(u) \cos(u_J) \sin(\Omega-\Omega_J) \cos(i)$$
$$+ \sin(u) \sin(u_J) \sin(\Omega-\Omega_J) \cos(i_J)$$
$$- \cos(u) \sin(u_J) \cos(\Omega-\Omega_J) \cos(i) \cos(i_J)$$
$$- \cos(u) \sin(u_J) \sin(i) \sin(i_J)) (\Delta^{-3} - r_J^{-3})$$

$$W = \sin(\Omega)\sin(i)\ m_J(r_J(\cos(u_J)\cos(\Omega_J)$$
$$- \sin(u_J)\sin(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3})$$
$$- (\cos(u)\cos(\Omega) - \sin(u)\sin(\Omega)\cos(i))\ r\,\Delta^{-3})$$
$$- \cos(\Omega)\sin(i)\ m_J(r_J(\cos(u_J)\sin(\Omega_J)$$
$$+ \sin(u_J)\cos(\Omega_J)\cos(i_J))(\Delta^{-3} - r_J^{-3})$$
$$- (\cos(u)\sin(\Omega) + \sin(u)\cos(\Omega)\cos(i))\ r\,\Delta^{-3})$$
$$+ \cos(i)\ m_J(r_J\ \sin(u_J)\sin(i_J)\ (\Delta^{-3} - r_J^{-3})$$
$$- \sin(u)\sin(i)\ r\ \Delta^{-3})$$

The coefficient of $m_J r_J(\Delta^{-3} - r_J^{-3})$ in this expression is:

$$\sin(\Omega)\ \sin(i)\ (\cos(u_J)\ \cos(\Omega_J) - \sin(u_J)\ \sin(\Omega_J)\ \cos(i_J))$$
$$- \cos(\Omega)\ \sin(i)\ (\cos(u_J)\ \sin(\Omega_J) + \sin(u_J)\ \cos(\Omega_J)\ \cos(i_J))$$
$$+ \cos(i)\ \sin(u_J)\ \sin(i_J)$$

$$= \cos(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\sin(i)$$
$$- \sin(u_J)\ \sin(\Omega)\ \sin(\Omega_J)\ \sin(i)\ \cos(i_J)$$
$$- \cos(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \sin(i)$$
$$- \sin(u_J)\ \cos(\Omega)\ \cos(\Omega_J)\ \sin(i)\ \cos(i_J)$$
$$+ \sin(u_J)\ \cos(i)\ \sin(i_J)$$

$$= \cos(u_J)\ \sin(\Omega-\Omega_J)\ \sin(i)$$
$$- \sin(u_J)\ \cos(\Omega-\Omega_J)\ \sin(i)\ \cos(i_J)$$
$$+ \sin(u_J)\ \cos(i)\ \sin(i_J)$$

The coefficient of $-m_J r \Delta^{-3}$ is:

$$\sin(\Omega)\ \sin(i)\ (\cos(u)\ \cos(\Omega) - \sin(u)\ \sin(\Omega)\ \cos(i))$$
$$- \cos(\Omega)\ \sin(i)\ (\cos(u)\ \sin(\Omega) + \sin(u)\ \cos(\Omega)\ \cos(i))$$
$$+ \cos(i)\ \sin(u)\ \sin(i)$$

$$= \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \sin(i)$$
$$- \sin(u)\ \sin^2(\Omega)\ \sin(i)\ \cos(i)$$
$$- \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \sin(i)$$
$$- \sin(u)\ \cos^2(\Omega)\ \sin(i)\ \cos(i)$$
$$+ \sin(u)\ \sin(i)\ \cos(i)$$

$$= 0$$

So the expression for W becomes:

$$W = m_J r_J (\cos(u_J)\ \sin(\Omega-\Omega_J)\ \sin(i)$$
$$- \sin(u_J)\ \cos(\Omega-\Omega_J)\ \sin(i)\ \cos(i_J)$$
$$+ \sin(u_J)\ \cos(i)\ \sin(i_J))\ (\Delta^{-3} - r_J^{-3})$$

Collecting these together, we thus have:

$$S = m_J(r_J(\cos(u)\cos(u_J)\cos(\Omega-\Omega_J)$$
$$-\sin(u)\cos(u_J)\sin(\Omega-\Omega_J)\cos(i)$$
$$+\cos(u)\sin(u_J)\sin(\Omega-\Omega_J)\cos(i_J)$$
$$+\sin(u)\sin(u_J)\cos(\Omega-\Omega_J)\cos(i)\cos(i_J)$$
$$+\sin(u)\sin(u_J)\sin(i)\sin(i_J))(\Delta^{-3} - r_J^{-3})$$
$$- r\Delta^{-3})$$

$$T = -m_J(r_J(\sin(u)\cos(u_J)\cos(\Omega-\Omega_J)$$
$$+\cos(u)\cos(u_J)\sin(\Omega-\Omega_J)\cos(i)$$
$$+\sin(u)\sin(u_J)\sin(\Omega-\Omega_J)\cos(i_J)$$
$$-\cos(u)\sin(u_J)\cos(\Omega-\Omega_J)\cos(i)\cos(i_J)$$
$$-\cos(u)\sin(u_J)\sin(i)\sin(i_J))(\Delta^{-3} - r_J^{-3})$$

$$W = m_J\, r_J(\cos(u_J)\sin(\Omega-\Omega_J)\sin(i)$$
$$-\sin(u_J)\cos(\Omega-\Omega_J)\sin(i)\cos(i_J)$$
$$+\sin(u_J)\cos(i)\sin(i_J))(\Delta^{-3} - r_J^{-3})$$

At this stage we can also produce an expression for calculating Δ, the distance between the minor planet and Jupiter. By simple geometry,

$$\Delta^2 = (x-x_J)^2 + (y-y_J)^2 + (z-z_J)^2$$

in any rectangular frame of reference (x,y,z). Using the ecliptic frame of reference as before, we have already determined that:

$$x = r(\cos(u)\ \cos(\Omega) - \sin(u)\ \sin(\Omega)\ \cos(i))$$
$$y = r(\cos(u)\ \sin(\Omega) + \sin(u)\ \cos(\Omega)\ \cos(i))$$
$$z = r\ \sin(u)\ \sin(i)$$

and

$$x_J = r_J(\cos(u_J)\ \cos(\Omega_J) - \sin(u_J)\ \sin(\Omega_J)\ \cos(i_J))$$
$$y_J = r_J(\cos(u_J)\ \sin(\Omega_J) + \sin(u_J)\ \cos(\Omega_J)\ \cos(i_J))$$
$$z_J = r_J\ \sin(u_J)\ \sin(i_J)$$

Substituting these into the expression for Δ^2, we have:

$$\begin{aligned}\Delta^2 = {} & (r\ \cos(u)\ \cos(\Omega) - r\ \sin(u)\ \sin(\Omega)\ \cos(i) \\ & \quad - r_J\ \cos(u_J)\ \cos(\Omega_J) + r_J\ \sin(u_J)\ \sin(\Omega_J)\ \cos(i_J)) \\ & + (r\ \cos(u)\ \sin(\Omega) + r\ \sin(u)\ \cos(\Omega)\ \cos(i) \\ & \quad - r_J\ \cos(u_J)\ \sin(\Omega_J) - r_J\ \sin(u_J)\ \cos(\Omega_J)\ \cos(i_J)) \\ & + (r\ \sin(u)\ \sin(i) - r_J\ \sin(u_J)\ \sin(i_J))\end{aligned}$$

$$
\begin{aligned}
= {} & r^2 \cos^2(u) \cos^2(\Omega) \\
& + r^2 \sin^2(u) \sin^2(\Omega) \cos^2(i) \\
& + r_J^2 \cos^2(u_J) \cos^2(\Omega_J) \\
& + r_J^2 \sin^2(u_J) \sin^2(\Omega_J) \cos^2(i_J) \\
& - 2 r^2 \sin(u) \cos(u) \sin(\Omega) \cos(\Omega) \cos(i) \\
& - 2 r r_J \cos(u) \cos(u_J) \cos(\Omega) \cos(\Omega_J) \\
& + 2 r r_J \cos(u) \sin(u_J) \cos(\Omega) \sin(\Omega_J) \cos(i_J) \\
& + 2 r r_J \sin(u) \cos(u_J) \sin(\Omega) \cos(\Omega_J) \cos(i) \\
& - 2 r r_J \sin(u) \sin(u_J) \sin(\Omega) \sin(\Omega_J) \cos(i) \cos(i_J) \\
& - 2 r_J^2 \sin(u_J) \cos(u_J) \sin(\Omega_J) \cos(\Omega_J) \cos(i_J) \\
& + r^2 \cos^2(u) \sin^2(\Omega) \\
& + r^2 \sin^2(u) \cos^2(\Omega) \cos^2(i) \\
& + r_J^2 \cos^2(u_J) \sin^2(\Omega_J) \\
& + r_J^2 \sin^2(u_J) \cos^2(\Omega_J) \cos^2(i_J) \\
& + 2 r^2 \sin(u) \cos(u) \sin(\Omega) \cos(\Omega) \cos(i) \\
& - 2 r r_J \cos(u) \cos(u_J) \sin(\Omega) \sin(\Omega_J) \\
& - 2 r r_J \cos(u) \sin(u_J) \sin(\Omega) \cos(\Omega_J) \cos(i_J) \\
& - 2 r r_J \sin(u) \cos(u_J) \cos(\Omega) \sin(\Omega_J) \cos(i) \\
& - 2 r r_J \sin(u) \sin(u_J) \cos(\Omega) \cos(\Omega_J) \cos(i) \cos(i_J) \\
& + 2 r_J^2 \sin(u_J) \cos(u_J) \sin(\Omega_J) \cos(\Omega_J) \cos(i_J) \\
& + r^2 \sin^2(u) \sin^2(i) \\
& + r_J^2 \sin^2(u_J) \sin^2(i_J) \\
& - 2 r r_J \sin(u) \sin(u_J) \sin(i) \sin(i_J)
\end{aligned}
$$

In this expression, the coefficient of r^2 is:

$$
\begin{aligned}
&\cos^2(u)\ \cos^2(\Omega) \\
&+ \sin^2(u)\ \sin^2(\Omega)\ \cos^2(i) \\
&- 2\ \sin(u)\ \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&+ \cos^2(u)\ \sin^2(\Omega) \\
&+ \sin^2(u)\ \cos^2(\Omega)\ \cos^2(i) \\
&+ 2\ \sin(u)\ \cos(u)\ \sin(\Omega)\ \cos(\Omega)\ \cos(i) \\
&+ \sin^2(u)\ \sin^2(i) \\
&= 1
\end{aligned}
$$

Similarly the coefficient of r_J^2 is:

$$
\begin{aligned}
&\cos^2(u_J)\ \cos^2(\Omega_J) \\
&+ \sin^2(u_J)\ \sin^2(\Omega_J)\ \cos^2(i_J) \\
&- 2\ \sin(u_J)\ \cos(u_J)\ \sin(\Omega_J)\ \cos(\Omega_J)\ \cos(i_J) \\
&+ \cos^2(u_J)\ \sin^2(\Omega_J) \\
&+ \sin^2(u_J)\ \cos^2(\Omega_J)\ \cos^2(i_J) \\
&+ 2\ \sin(u_J)\ \cos(u_J)\ \sin(\Omega_J)\ \cos(\Omega_J)\ \cos(i_J) \\
&+ \sin^2(u_J)\ \sin^2(i_J) \\
&= 1
\end{aligned}
$$

The coefficient of $2\ r\ r_J$ is:

$$
\begin{aligned}
&- \cos(u)\ \cos(u_J)\ \cos(\Omega)\ \cos(\Omega_J) \\
&+ \cos(u)\ \sin(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i_J) \\
&+ \sin(u)\ \cos(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i) \\
&- \sin(u)\ \sin(u_J)\ \sin(\Omega)\ \sin(\Omega_J)\ \cos(i)\ \cos(i_J) \\
&- \cos(u)\ \cos(u_J)\ \sin(\Omega)\ \sin(\Omega_J) \\
&- \cos(u)\ \sin(u_J)\ \sin(\Omega)\ \cos(\Omega_J)\ \cos(i_J) \\
&- \sin(u)\ \cos(u_J)\ \cos(\Omega)\ \sin(\Omega_J)\ \cos(i) \\
&- \sin(u)\ \sin(u_J)\ \cos(\Omega)\ \cos(\Omega_J)\ \cos(i)\ \cos(i_J) \\
&- \sin(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)
\end{aligned}
$$

$$
\begin{aligned}
= &- \cos(u)\ \cos(u_J)\ \cos(\Omega-\Omega_J) \\
&+ \sin(u)\ \cos(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i) \\
&- \cos(u)\ \sin(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i_J) \\
&- \sin(u)\ \sin(u_J)\ \cos(\Omega-\Omega_J)\ \cos(i)\ \cos(i_J) \\
&- \sin(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)
\end{aligned}
$$

So we have:

$$
\begin{aligned}
\Delta^2 = r^2 + r_J^2 - 2\ r\ r_J(&\cos(u)\cos(u_J)\cos(\Omega-\Omega_J) \\
&- \sin(u)\cos(u_J)\sin(\Omega-\Omega_J)\cos(i) \\
&+ \cos(u)\sin(u_J)\sin(\Omega-\Omega_J)\cos(i_J) \\
&+ \sin(u)\sin(u_J)\cos(\Omega-\Omega_J)\cos(i)\cos(i_J) \\
&+ \sin(u)\sin(u_J)\sin(i)\sin(i_J))
\end{aligned}
$$

It may be remarked that these rather lengthy expressions for S, T, W and Δ^2 were reached by calculations using the ecliptic as the reference plane. It would have been equally possible to use the plane of the orbit, either of Jupiter, or of the minor planet, as the reference plane. In the first case, the equations for the coordinates of Jupiter would have been much simpler, and those for the coordinates of the minor planet much more complex; in the second case, vice versa. In either case, the end results would have been the same as those obtained here. The use of the ecliptic as the reference plane introduces a degree of symmetry into the calculations, which makes it preferable. The normal approach in treating the restricted case of the three-body problem is to use, as the reference plane, the plane in which the two major bodies move that is, in this case, the orbital plane of Jupiter. However, departure from this principle makes the calculations simpler without altering the final equations of motion.

Let us now pause to consider these expressions for S, T and W, for Δ^2, and for the derivatives of the orbital elements, to see what forms they would take if both i and i_J were zero. In this case both Ω and Ω_J would be indeterminate, and the problem would be entirely restricted to two dimensions. This is the situation considered by Schubart (1968).

Let us first take the expression which occurs in the formulae for both S and Δ^2, namely:

$$\begin{aligned} &\cos(u)\ \cos(u_J)\ \cos(\Omega-\Omega_J) \\ &- \sin(u)\ \cos(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i) \\ &+ \cos(u)\ \sin(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i_J) \qquad\qquad (3) \\ &+ \sin(u)\ \sin(u_J)\ \cos(\Omega-\Omega_J)\ \cos(i)\ \cos(i_J) \\ &+ \sin(u)\ \sin(u_J)\ \sin(i)\ \sin(i\ ) \end{aligned}$$

If $i = i_J = 0$, this reduces to:

$$\cos(\Omega - \Omega_J + u - u_J)$$

But the angle u was originally defined as

$$\begin{aligned} u &= \varpi - \Omega + v \\ &= \lambda - \Omega \end{aligned}$$

where $\lambda = v+\varpi$ is the true longitude of the minor planet. Similarly we may write

$$u_J = \lambda_J - \Omega_J$$

where λ_J is the true longitude of Jupiter.

So the expression (3) finally reduces to

$$\cos(\lambda-\lambda_J)$$

Substituting this into the formula for Δ^2, we have now:

$$\Delta^2 = r^2 + r_J^2 - 2\ r\ r_J\ \cos(\lambda-\lambda_J)$$

which is exactly what we would expect from geometrical consideration of the two-dimensional situation.
We now also find that the formula for S becomes:

$$S = m_J(r_J\ \cos(\lambda-\lambda_J)\ (\Delta^{-3} - r_J^{-3}) - r\ \Delta^{-3})$$

Let us consider now the expression

$$\begin{aligned}
&\sin(u)\ \cos(u_J)\ \cos(\Omega-\Omega_J)\\
&+ \cos(u)\ \cos(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i)\\
&+ \sin(u)\ \sin(u_J)\ \sin(\Omega-\Omega_J)\ \cos(i_J)\\
&- \cos(u)\ \sin(u_J)\ \cos(\Omega-\Omega_J)\ \cos(i)\ \cos(i_J)\\
&- \cos(u)\ \sin(u_J)\ \sin(i)\ \sin(i_J)
\end{aligned}$$

which occurs in the formula for T. If $i = i_J = 0$, this reduces to:

$$\sin(u - u_J + \Omega - \Omega_J)$$

and hence to $\sin(\lambda-\lambda_J)$.

The expression for T thus becomes:

$$T = -\ m_J\ r_J\ \sin(\lambda-\lambda_J)\ (\Delta^{-3} - r_J^{-3})$$

In the case of W, every term in the expression contains either $\sin(i)$ or $\sin(i_{\mathcal{J}})$, so if $i = i_{\mathcal{J}} = 0$ we have simply

$$W = 0$$

We turn finally to the equations for the derivatives of the six orbital elements, G, μ, ψ_1, ψ_2, χ_1 and χ_2. If we take S_J to be the reduced form of S (above) multiplied by G, T_J to be the reduced form of T multiplied by G, and W_J to be zero, these equations become:

$$\frac{dG}{dt} = \frac{r}{G} T_J$$

$$\frac{d\mu}{dt} = a^{-1.5} - \frac{p+q}{p} n_J - S_J \left(2 \frac{r}{G^2} \sqrt{(1-e^2)} + \frac{e \cos(v)}{1+\sqrt{(1-e^2)}} \right)$$

$$+ T_J \left(\frac{r}{G^2} + 1 \right) \frac{e \sin(v)}{1+\sqrt{(1-e^2)}}$$

$$\frac{d\psi_1}{dt} = S_J \sin(\lambda) + T_J \left(\cos(\lambda) \left(\frac{r}{G^2} + 1 \right) + \frac{r}{G^2} \psi_2 \right)$$

$$\frac{d\psi_2}{dt} = - S_J \cos(\lambda) + T_J \left(\sin(\lambda) \left(\frac{r}{G^2} + 1 \right) + \frac{r}{G^2} \psi_1 \right)$$

$$\frac{d\chi_1}{dt} = \frac{d\chi_2}{dt} = 0$$

These simplified equations for S and T, for Δ^2, and for the derivatives of the four orbital elements G, μ, ψ_1 and ψ_2 are identical with those given by Schubart (1968). If the inclination of the minor planet is zero, then $\chi_1 = \chi_2 = 0$, and since the derivatives of these elements also reduce to zero, they remain zero for all time, as we should expect.

At each step of the integration, the orbital parameters G, μ, ψ_1, ψ_2, χ_1 and χ_2 were known, from the previous step. From these, all the necessary variables could be calculated, as follows.

The mean longitude of Jupiter was assumed to increase steadily at the rate of n_J, from its starting-value; n_J was calculated as $\sqrt{(1 + m_J)}$, the mass of Jupiter being taken as $(1047.355)^{-1}$ solar masses, and the starting-value of the longitude was found from the "Astronomical Ephemeris" for the date of the beginning of the integration. The mean longitude of the minor planet could then be found from the current value of μ, by the formula

$$l = \mu + \frac{p+q}{p} l_J$$

The mean anomaly of Jupiter was equal to its mean longitude, since longitudes were measured from the perihelion of Jupiter. The mean anomaly of the minor planet was found by subtracting the current value of ϖ from its mean longitude. $(\tan(\varpi) = \psi_2/\psi_1.)$

Kepler's equation was then solved twice to find the true anomalies and the radius vectors of the two planets. For Jupiter, the eccentricity was assumed constant at 0.048 for all the integrations; for the minor planet the current value of the eccentricity was used, calculated from :

$$e = \sqrt{\psi_1^2 + \psi_2^2}$$

The semi-major axis of Jupiter's orbit was always equal to 1; for the minor planet, it was calculated from

$$a = \frac{G^2}{1-e^2}$$

No special technique was used to speed up the solution of Kepler's equation; the process was simply to take the mean anomaly M as a first approximation E_1 to the eccentric anomaly, and then calculate

$$E_2 = M + e \sin(E_1)$$

for successive approximations, until the results converged to a constant value E. The eccentricities used were all so small that only a few iterations were needed in each case.

The true anomaly v was then calculated from

$$\tan(v) = \frac{\sqrt{1-e^2}\,\sin(E)}{\cos(E) - e}$$

and the radius vector r from

$$r = a\,(\,1 - e\cos(E)\,)$$

The true longitude λ_J of Jupiter was the same as the true anomaly, and the true longitude λ of the minor planet was found by adding ϖ to the true anomaly v.

The values for the inclination and the longitude of the ascending node of Jupiter were held constant throughout the integration, their values being found from the "Astronomical Ephemeris" for the starting-date. The corresponding values for the minor planet were calculated from the current values of χ_1 and χ_2, by

$\sin(i) = \sqrt{\chi_1^2 + \chi_2^2}$

$\tan(\Omega) = \chi_2 / \chi_1$

Thus all the variables needed to calculate S, T and W, and consequently to calculate the derivatives of the six orbital elements, could be found.

It was now necessary to apply the averaging technique (described by Schubart, 1968) at each step in the integration. The mean longitudes of the two planets were allowed to increase steadily through a circuit of p orbits of Jupiter, or p+q orbits of the minor planet; all the other orbital parameters were kept constant. At intervals around the circuit, the derivatives of the six orbital elements were calculated and stored. The number of intervals varied according to the values of p and q in use; it was defined to be 72*(p+q), or, in other words, each interval corresponded to a movement of 5 degrees in the mean longitude of the minor planet. The stored values of the derivative were then averaged, using a Newton-Cotes formula: if the N values of one of the derivatives were $x_1, \ldots, x_N$, then the average value was taken to be

$$x = (x_1 + 4\ x_2 + 2\ x_3 + 4\ x_4 + \ldots$$
$$\ldots + 2\ x_{N-2} + 4\ x_{N-1} + x_N)\ /\ 3\ N$$

In fact the entire process of calculating and averaging the derivatives was carried out in a subroutine of the main program. At each step in the main integration, the current values of the six orbital elements were passed to the subroutine, along with such constants as were necessary, and the averaged values of the derivatives of the six elements were returned. The main program then proceeded to carry out the next step in the integration of the orbital elements; the procedures used for this will be described next.

The main part of the integration was carried out by an Adams "predictor-corrector" method. This technique uses the value of the required function y_n and its derivative y'_n at the current point, and the derivatives y'_{n-1}, y'_{n-2} etc. at several previous points, to "predict" the value y_{n+1} at the next point, by an expression of the form:

$$y_{n+1} = y_n + h\,(A_p y'_n + B_p y'_{n-1} + C_p y'_{n-2} + \ldots)$$

where h is the step-length of the integration, A_p, B_p etc. are constants, and the expression is carried to as many terms as required.

From this "predicted" value of y_{n+1} the derivative y'_{n+1} is calculated, and the value of y_{n+1} is then "corrected" by the formula:

$$y_{n+1} = y_n + h\,(A_c y'_{n+1} + B_c y'_n + C_c y'_{n-1} + \ldots)$$

where A_c, B_c etc. are further constants, and the expression is carried to the same number of terms as in the "predictor" formula.

This "corrected" value of y_{n+1} is accepted as the value for the next point, and a "corrected" value of y'_{n+1} is calculated from it, for use in the subsequent stages of the integration.

It would, of course, be possible to apply the "corrector" formula more than once, but this does not necessarily improve the accuracy of the method. It is generally considered better either to decrease the step-length, or to take more terms in the formulae, or both. Before the full integration was carried out, therefore, a series of trial calculations was made, with different step-lengths and different numbers of terms, to see which combination gave the greatest accuracy without using excessive computing-time. The function integrated was one whose integral was known analytically, namely $y' = \sin(x)$. The accuracy of the various sets of results was compared, and the final choice was a step-length of 0.5 time-units, together with six terms in each of the formulae.

(As explained in on page 89, a time-unit is about 1.9 years, so the step-length here is nearly one year. Schubart (1968) suggests an interval of 1.5 or 2 years. It is the averaging-out of the short-period effects that enables such long intervals to be used, with the result that the orbit can be studied over a much longer period of time than would otherwise be feasible.)

To carry out the trial calculations, it was of course necessary first to evaluate the constants A_p, B_p,... and A_c, B_c,... in the Adams formulae. The method for this consists in making the n-term formulae exactly correct for the integration of an n'th-order polynomial. We give here, in detail, the method of determining the constants for the six-term formulae which were eventually used in the main integration of the planet orbits.

We assume that the function to be integrated is of the form:

$$y_n = a + b\,x_n + c\,x_n^2 + d\,x_n^3 + e\,x_n^4 + f\,x_n^5 + g\,x_n^6$$

We write the "predictor" formula as:

$$(y_{n+1} - y_n)/h = A_p y'_n + B_p y'_{n-1} + C_p y'_{n-2} + D_p y'_{n-3} + E_p y'_{n-4} + F_p y'_{n-5} \qquad (1)$$

The left-hand side of this equation can be expanded as:

$$(a + b(x_n+h) + c(x_n+h)^2 + e(x_n+h)^3 + f(x_n+h)^4 + g(x_n+h)^5 - a - b\,x_n - c\,x_n^2 - d\,x_n^3 - e\,x_n^4 - f\,x_n^5 - g\,x_n^6)/h$$

which can be rearranged as:

$$b + c(2x_n + h) + d(3x_n^2 + 3x_n h + h^2)$$
$$+ e(4x_n^3 + 6x_n^2 h + 4x_n h^2 + h^3)$$
$$+ f(5x_n^4 + 10x_n^3 h + 10x_n^2 h^2 + 5x_n h^3 + h^4)$$
$$+ g(6x_n^5 + 15x_n^4 h + 20x_n^3 h^2 + 15x_n^2 h^3 + 6x_n h^4 + h^5)$$

To evaluate the right-hand side of equation (1) we need the derivatives y'. Differentiating y_n gives:

$$y'_n = b + 2cx_n + 3dx_n^2 + 4ex_n^3 + 5fx_n^4 + 6gx_n^5$$

and consequently the other derivatives are:

$$y'_{n-1} = b + 2c(x_n-h) + 3d(x_n-h)^2 + 4e(x_n-h)^3 + 5f(x_n-h)^4 + 6g(x_n-h)^5$$

$$y'_{n-2} = b + 2c(x_n-2h) + 3d(x_n-2h)^2 + 4e(x_n-2h)^3 + 5f(x_n-2h)^4 + 6g(x_n-2h)^5$$

$$y'_{n-3} = b + 2c(x_n-3h) + 3d(x_n-3h)^2 + 4e(x_n-3h)^3 + 5f(x_n-3h)^4 + 6g(x_n-3h)^5$$

$$y'_{n-4} = b + 2c(x_n-4h) + 3d(x_n-4h)^2 + 4e(x_n-4h)^3 + 5f(x_n-4h)^4 + 6g(x_n-4h)^5$$

$$y'_{n-5} = b + 2c(x_n-5h) + 3d(x_n-5h)^2 + 4e(x_n-5h)^3 + 5f(x_n-5h)^4 + 6g(x_n-5h)^5$$

These expressions are substituted into the right-hand side of equation (1), and the coefficients of each term on the left- and right-hand sides are equated (i.e. the coefficients of b, cx_n, ch, dx_n^2 etc.). This gives a total of 21 equations in A_p, B_p, ..., F_p. However, only six of these are independent, the remainder being multiples of them. The six, in their simplest forms, are obtained by equating the coefficients of b, ch, dh^2, eh^3, fh^4 and gh^5; they are:

$$1 = A_p + B_p + C_p + D_p + E_p + F_p$$

$$1 = -2B_p - 4C_p - 6D_p - 8E_p - 10F_p$$

$$1 = 3B_p + 12C_p + 27D_p + 48E_p + 75F_p$$

$$1 = -4B_p - 32C_p - 108D_p - 256E_p - 500F_p$$

$$1 = 5B_p + 80C_p + 405D_p + 1280E_p + 3125F_p$$

$$1 = -6B_p - 192C_p - 1458D_p - 6144E_p - 18750F_p$$

The solution of these six simultaneous linear equations may be expressed, for convenience, with a common denominator, as:

$$A_p = \frac{4277}{1440} \qquad B_p = -\frac{7923}{1440} \qquad C_p = \frac{9982}{1440}$$

$$D_p = -\frac{7298}{1440} \qquad E_p = \frac{2877}{1440} \qquad F_p = -\frac{475}{1440}$$

The coefficients of the "corrector" formula are evaluated in the same way. Writing the formula as:

$$(y_{n+1} - y_n)/h = A_c y'_{n+1} + B_c y'_n + C_c y'_{n-1} + D_c y'_{n-2} + E_c y'_{n-3} + F_c y'_{n-4} \qquad (2)$$

we see that the left-hand side is the same as in equation (1). To expand the right-hand side we need one extra derivative:

$$y'_{n+1} = b + 2c(x_n+h) + 3d(x_n+h)^2 + 4e(x_n+h)^3 + 5f(x_n+h)^4 + 6g(x_n+h)^5$$

which is added to the previous table of derivatives.

Equating coefficients, as before, on either side of equation (2) gives again 21 equations in A_c, B_c, ..., F_c, of which only six are independent. These six, in their simplest forms, are:

$$1 = A_c + B_c + C_c + D_c + E_c + F_c$$

$$1 = 2A_c - 2C_c - 4D_c - 6E_c - 8F_c$$

$$1 = 3A_c + 3C_c + 12D_c + 27E_c + 48F_c$$

$$1 = 4A_c - 4C_c - 32D_c - 108E_c - 256F_c$$

$$1 = 5A_c + 5C_c + 80D_c + 405E_c + 1280F_c$$

$$1 = 6A_c - 6C_c - 192D_c - 1458E_c - 6144F_c$$

The solutions of these equations may be expressed, with the same common denominator as before, as:

$$A_c = \frac{475}{1440} \qquad B_c = \frac{1427}{1440} \qquad C_c = -\frac{798}{1440}$$

$$D_c = \frac{482}{1440} \qquad E_c = -\frac{173}{1440} \qquad F_c = \frac{27}{1440}$$

So the two formulae for the six-term version of the Adams method are:

predictor:

$$y_{n+1} = y_n + h\,(4277y'_n - 7923y'_{n-1} + 9982y'_{n-2} - 7298y'_{n-3} + 2877y'_{n-4} - 475y'_{n-5})\,/1440$$

corrector:

$$y_{n+1} = y_n + h\,(475y'_{n+1} + 1427y'_n - 798y'_{n-1} + 482y'_{n-2} - 173y'_{n-3} + 27y'_{n-4})\,/1440$$

The foregoing description of the Adams method refers to a single function y(x). In the case of the orbit of a minor planet, we are dealing with six variables, the six elements of the orbit, and the derivative of each is a function, not only of the time, but also of all the six elements. However, the Adams method can easily be extended to this situation, by using the values of all the variables to calculate each of the derivatives, at each stage.

Obviously, the great advantage of the Adams method is that each calculated derivative is used several times over, and that, at each step, only two extra derivatives of each function need to be calculated (the "predicted" and "corrected" values of y'_{n+1}). In practice, a 6 * 6 array was used in the computer to store the derivatives of the six variables at six consecutive points. These were used in the "predictor" formula to "predict" the values of the six variables at the next point, and from these the values of the derivatives at the next point were also "predicted". The earliest set of derivatives was then dropped from the array, the remaining sets moved along, and the new set added. This new array was then used in the "corrector" formula to give corrected values of the six variables at the next point; these values were stored. The "corrected" values of the derivatives at the next point were then calculated, and put into the array in place of the "predicted" ones. The array was now ready for the next integration step.

Owing to the length of computing-time required to carry out a long integration, it was not generally possible to complete it in one uninterrupted period. The program was therefore designed to run in several stages. The values of the six variables at each integration step were stored on magnetic disc as they were calculated, and so could be retrieved by the program when it re-started after an interruption. The values of the derivatives in the 6 * 6 array were, however, lost when the program was interrupted. These were therefore re-calculated from the values of the variables at the last six points, and so the array was re-filled and the integration re-started.

Obviously this technique could not be used at the very beginning of an integration, when only the starting-values of the six variables were known, without any values at previous points. A different method was therefore used to set up each integration, based on the Runge-Kutta method of numerical integration.

In its simplest form, the Runge-Kutta method investigates a function $x(t)$, with known derivative $x' = f(x,t)$. Given the value of x at a starting-point t, the value of x at the next point t+h is calculated by computing the intermediate quantities k_1, k_2, k_3 and k_4. These are:

$$k_1 = f(x(t), t) * h$$

$$k_2 = f(x(t)+0.5k_1, t+0.5h) * h$$

$$k_3 = f(x(t)+0.5k_2, t+0.5h) * h$$

$$k_4 = f(x(t)+k_3, t+h) * h$$

The value of x(t+h) is then calculated from:

$$x(t+h) = x(t) + (k_1 + 2k_2 + 2k_3 + k_4) /6$$

This simple one-dimensional situation can be extended to include several mutually dependent functions. In this investigation we are dealing with six, the six elements of the orbit of a minor planet. For simplicity, at this stage we will write them as $x_1, \ldots, x_6$. Each function has a known derivative depending on the values of all six elements and on time. We may write this schematically as:

$$x_1' = f_1(x_1, x_2, x_3, x_4, x_5, x_6, t)$$
$$\vdots$$
$$x_6' = f_6(x_1, x_2, x_3, x_4, x_5, x_6, t)$$

where x_1 is taken to mean $x_1(t)$, etc. The Runge-Kutta method, appled to these equations, gives six equations for k_1, six for k_2 and so on. Writing only the first and last of each set, they are:

$$k_{1,1} = f_1(x_1, x_2, x_3, x_4, x_5, x_6, t) * h$$

$$\vdots$$

$$k_{1,6} = f_6(x_1, x_2, x_3, x_4, x_5, x_6, t) * h$$

$$k_{2,1} = f_1(x_1 + \tfrac{1}{2}k_{1,1}, x_2 + \tfrac{1}{2}k_{1,2}, x_3 + \tfrac{1}{2}k_{1,3}, x_4 + \tfrac{1}{2}k_{1,4}, x_5 + \tfrac{1}{2}k_{1,5}, x_6 + \tfrac{1}{2}k_{1,6}, t + \tfrac{1}{2}h) * h$$

$$\vdots$$

$$k_{2,6} = f_6(x_1 + \tfrac{1}{2}k_{1,1}, x_2 + \tfrac{1}{2}k_{1,2}, x_3 + \tfrac{1}{2}k_{1,3}, x_4 + \tfrac{1}{2}k_{1,4}, x_5 + \tfrac{1}{2}k_{1,5}, x_6 + \tfrac{1}{2}k_{1,6}, t + \tfrac{1}{2}h) * h$$

$$k_{3,1} = f_1(x_1 + \tfrac{1}{2}k_{2,1}, x_2 + \tfrac{1}{2}k_{2,2}, x_3 + \tfrac{1}{2}k_{2,3}, x_4 + \tfrac{1}{2}k_{2,4}, x_5 + \tfrac{1}{2}k_{2,5}, x_6 + \tfrac{1}{2}k_{2,6}, t + \tfrac{1}{2}h) * h$$

$$\vdots$$

$$k_{3,6} = f_6(x_1 + \tfrac{1}{2}k_{2,1}, x_2 + \tfrac{1}{2}k_{2,2}, x_3 + \tfrac{1}{2}k_{2,3}, x_4 + \tfrac{1}{2}k_{2,4}, x_5 + \tfrac{1}{2}k_{2,5}, x_6 + \tfrac{1}{2}k_{2,5}, t + \tfrac{1}{2}h) * h$$

$$k_{4,1} = f_1(x_1 + k_{3,1}, x_2 + k_{3,2}, x_3 + k_{3,3}, x_4 + k_{3,4}, x_5 + k_{3,5}, x_6 + k_{3,6}, t + h) * h$$

$$\vdots$$

$$k_{4,6} = f_6(x_1 + k_{3,1}, x_2 + k_{3,2}, x_3 + k_{3,3}, x_4 + k_{3,4}, x_5 + k_{3,5}, x_6 + k_{3,6}, t + h) * h$$

We can then calculate $x_1(t+h)$, $x_2(t+h)$ etc. by:

$$x_1(t+h) = x_1(t) + (k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})/6$$

$$\vdots$$

$$x_6(t+h) = x_6(t) + (k_{1,6} + 2k_{2,6} + 2k_{3,6} + k_{4,6})/6$$

A setting-up program was therefore written, based on this method, to be used each time the orbit of a minor planet was to be integrated. The conventional elements of the orbit were fed in, either from a published orbit, or from an orbit derived directly from observations. The program converted these into the computational elements G, μ, ψ_1, ψ_2, χ_1 and χ_2, and set the time as zero. The step-length h was taken as 0.5 time-units, as in the main integration program. The same equations for the derivatives, and the same averaging technique, were used. The program used the Runge-Kutta technique to calculate the values of the six elements at t = 0.5, which it stored; it then used these values as starting-points to calculate the values at t = 1.0, and the process was repeated until sufficient sets of values had been stored to enable the main integration program, using the Adams method, to be run.

Once again, preliminary calculations were carried out to investigate the accuracy of the Runge-Kutta method of integration, and it was found to be comparable with the Adams method in the main program, for the same step-size. However, the Runge-Kutta method is far more wasteful of computing-time, since none of the intermediate values of the derivatives used to calculate the k's can be re-used, and the k's themselves have to be calculated afresh at every step. For this reason, the Runge-Kutta method was only used at the very beginning of the integration, where the Adams method could not be applied.

Details of the Adams and Runge-Kutta methods of integration may be found in Hamming (1973).

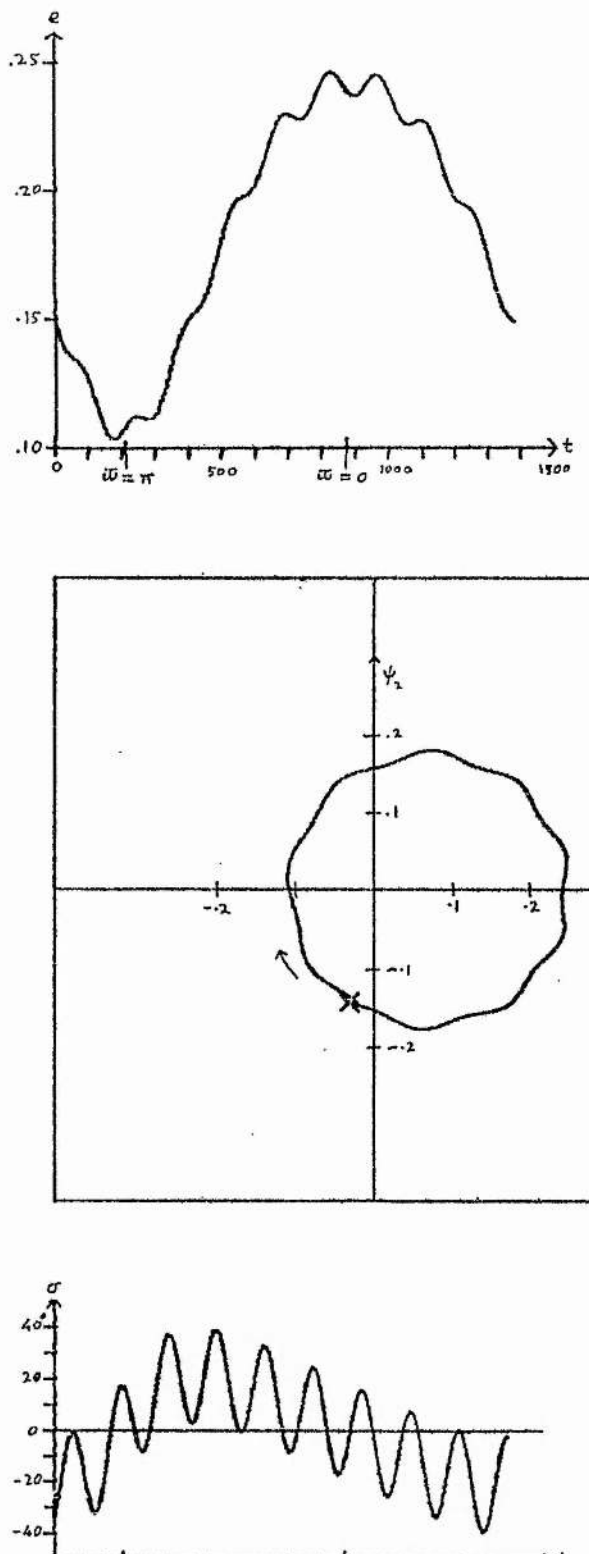
The first stage of checking the correctness of the equations and the accuracy of the numerical methods was to try to reproduce the results of Schubart (1968). For this purpose, an integration was carried out on (153) Hilda, using the same starting-values as given in his paper, and setting the inclination both of (153) Hilda and of Jupiter to zero.

The resultant changes in the orbit were plotted out in three graphs. The first simply showed the behaviour of the eccentricity as a function of the time; the second plotted the variable ψ_2 against ψ_1, giving a polar diagram with radius corresponding to the eccentricity and angle corresponding to the longitude of perihelion. The third graph plotted the critical angle σ against time, where

$$\sigma = -(\varpi + \mu \frac{p}{q})$$

This angle is defined by Schubart as a quantity analogous to μ.

The resultant graphs were scaled down to the same size and proportions as published in Schubart's paper, and were found to agree closely. These reduced graphs are given here as diagram 8.



RESULTS OF INTEGRATION

OF ORBIT OF (153) HILDA

Diagram 8

A second check was made on the stability of the numerical methods used, over long-period integrations. For this, an integration was carried out on the orbit of (909) Ulla, taking the ratio of mean motion to that of Jupiter as 9/5, but setting the mass of Jupiter to zero. This meant that all perturbing forces were zero, and consequently that the orbit should remain unchanged indefinitely. Any changes in the orbit could therefore be attributed to numerical instability, such as is inevitably caused by an accumulation of rounding-errors when an integration step is repeated many times.

The integration was carried out over 1507 time-units, or over 3000 steps. At the end of this period, the largest change in an orbital parameter was in the longitude of perihelion, which had drifted by 0.0007 degrees. The longitude of the ascending node had changed by only one-tenth as much. The eccentricity changed by less than 0.000001, the inclination by 0.00005 degrees, and the semi-major axis by 0.000004 A.U.

These changes are all much less than the effects observed in the orbital parameters when the mass of Jupiter was given its true value. Consequently it was assumed that the numerical methods used were acceptable.

The methods described in the preceding pages were used to integrate the orbits of the planets which had already been selected for observation. These were: (87) Sylvia; (107) Camilla; (334) Chicago; (414) Liriope; (909) Ulla; and (1574) Meyer. In addition, an integration was carried out on the orbit of (153) Hilda, taking into account the inclinations of both the minor planet and Jupiter, for comparison with the earlier "two-dimensional" integration.

In the cases of (153) Hilda and (334) Chicago, the ratio of mean motions was assumed to be 3/2; that is, p and q were taken as 2 and 1. For all the other planets, the ratio was taken as 9/5, giving p and q as 5 and 4. In addition, an extra integration was carried out on (909) Ulla, taking the ratio as 7/4 (p = 4, q = 3); the true ratio is actually nearer this value, and it was possible to compare the results of the two integrations on the same data.

In all the integrations carried out, the starting data were the orbits given in the "Ephemerides of Minor Planets", together with such information from the "Astronomical Ephemeris" as was required for the orbit of Jupiter. The orbit used for (909) Ulla was that given for 1962, as the orbit for 1968 was not available when the investigation began.

The output from the integration program, described in the previous section, was stored on magnetic disc, and also printed out. Each entry consisted of the time at the beginning of an integration step, followed by the values of the six orbital parameters at that step. (These were, of course, those orbital parameters used in the integration - G, μ, ψ_1, ψ_2, χ_1 and χ_2.) One further piece of information was printed at this stage: a warning message in the event of the minor planet suffering a close approach to Jupiter.

The necessity of ascertaining whether a close approach has occurred arises from the use of an averaging process. The averaging of the short-period terms is carried out at a fixed number of equidistant points around a circuit of p+q orbits of the minor planet, or p orbits of Jupiter. If at any point in this circuit the minor planet should approach very close to Jupiter, the gravitational force will be much greater at that point, and will change rapidly on either side of it. The averaging process, which merely samples the force at intervals, may miss these large values and consequently give an inaccurate result. For this reason it is important to ascertain if close approaches between the minor planet and Jupiter actually occur.

Schubart (1968) defines a quantity

$$\sigma = -(\tilde{\omega} + (p/q)\mu)$$

in order to investigate this possibility. A close approach is likely if conjunction between the minor planet and Jupiter occurs when the minor planet is at aphelion. (The eccentricity of Jupiter is so small that the position of Jupiter in its orbit has little effect.) Since

$\sigma = M - (l-l_J)(p+q)/q$

the conditions defined above, $l = l_J$ and $M = 180$ degrees, mean that $\sigma = 180$ degrees. Schubart therefore seeks to ensure that σ never comes near to 180 degrees, to show that no close approaches occur.

However, it is clear that there are other values of M and $l-l_J$ which may also make $\sigma = 180$ degrees, without necessarily meaning that a close approach has occurred. (For example, in the case where $(p+q)/p = 9/5$, the minor planet could be at perihelion, with a difference in longitude of 80 degrees between it and Jupiter.) In fact, it is to be expected (Schweizer, 1974) that σ will only librate between fixed values if the minor planet is exactly at a commensurability. Therefore, if it is found, as it was for (909) Ulla at an early stage of this investigation, that σ changes monotonically with time, this does not mean that a close approach must occur every time it passes through 180 degrees.

Investigation of σ was therefore rejected in favour of a more direct approach. The distance Δ between the minor planet and Jupiter is calculated explicitly at each step of the averaging process, in order to calculate the disturbing forces. The program was therefore designed to take note of the minimum value of Δ occurring during the entire averaging circuit. If this was less than a predetermined value, a warning message was printed out.

For minor planets with mean motions in a ratio of 9/5 with that of Jupiter, the threshold value was set at 0.25 in the units used, which is approximately 1.3 A.U. In the case of (909) Ulla, it was found that approaches closer than this occurred for about 9 time-units at intervals of about 44 time-units. However, over a prolonged period, the occurrences grew shorter and finally disappeared altogether, as the slowly-changing parameters of the orbit gradually made close approaches impossible. The smallest value of Δ detected was 0.2445, or about 1.27 A.U. When the orbit was re-integrated with values of p and q corresponding to a ratio of mean motions of 7/4, the threshold of 0.25 units was never crossed. Thus the danger of close approaches can be discounted in the case of (909) Ulla. In the integrations of the orbits of (87) Sylvia, (107) Camilla, (414) Liriope and (1574) Meyer, the threshold of 0.25 units was again never crossed, and so here too no close approaches need to be considered.

For (334) Chicago, the threshold had to be changed, because this planet is sufficiently close to Jupiter as to give a "close approach", by this definition, in every cycle. It was therefore re-set to 0.21 units (about 1.1 A.U.), and it was found that no approaches closer than this occurred throughout the integration. In the case of (153) Hilda, the threshold was lowered still further, to 0.12 units (about 0.6 A.U.), and this proved to be less than any distance occurring during the integration. In these latter two cases, this definition of "close" approach may be open to question, but it can be definitely stated that these two planets remain further from Jupiter than the distances stated, throughout the changes in their orbits described by the integration process.

Thus the immediate output from an integration was a computer listing of the six computational orbital parameters, together with warning messages of any close approaches. The next step was to convert these into the conventional orbital parameters; for this, a separate program was used.

This program returned to the original results of the integration, stored on magnetic disc. It computed the conventional parameters a, e, ϖ, i, and Ω for each step. (The mean anomaly of the planet was not computed, being so rapidly-changing as to be meaningless in this context; it could be derived from μ if particularly required at any point.) The formulae used were:

$$e = \sqrt{\psi_1^2 + \psi_2^2}$$

$$a = \frac{G^2}{1 - e^2}$$

$$\varpi = \tan^{-1}(\psi_2/\psi_1)$$

$$i = \sin^{-1}(\sqrt{\chi_1^2 + \chi_2^2})$$

$$\Omega = \tan^{-1}(\chi_2/\chi_1)$$

The results were then listed, giving at every step both the computational and the conventional orbital parameters. (The critical angle σ was also calculated and listed, for completeness.)

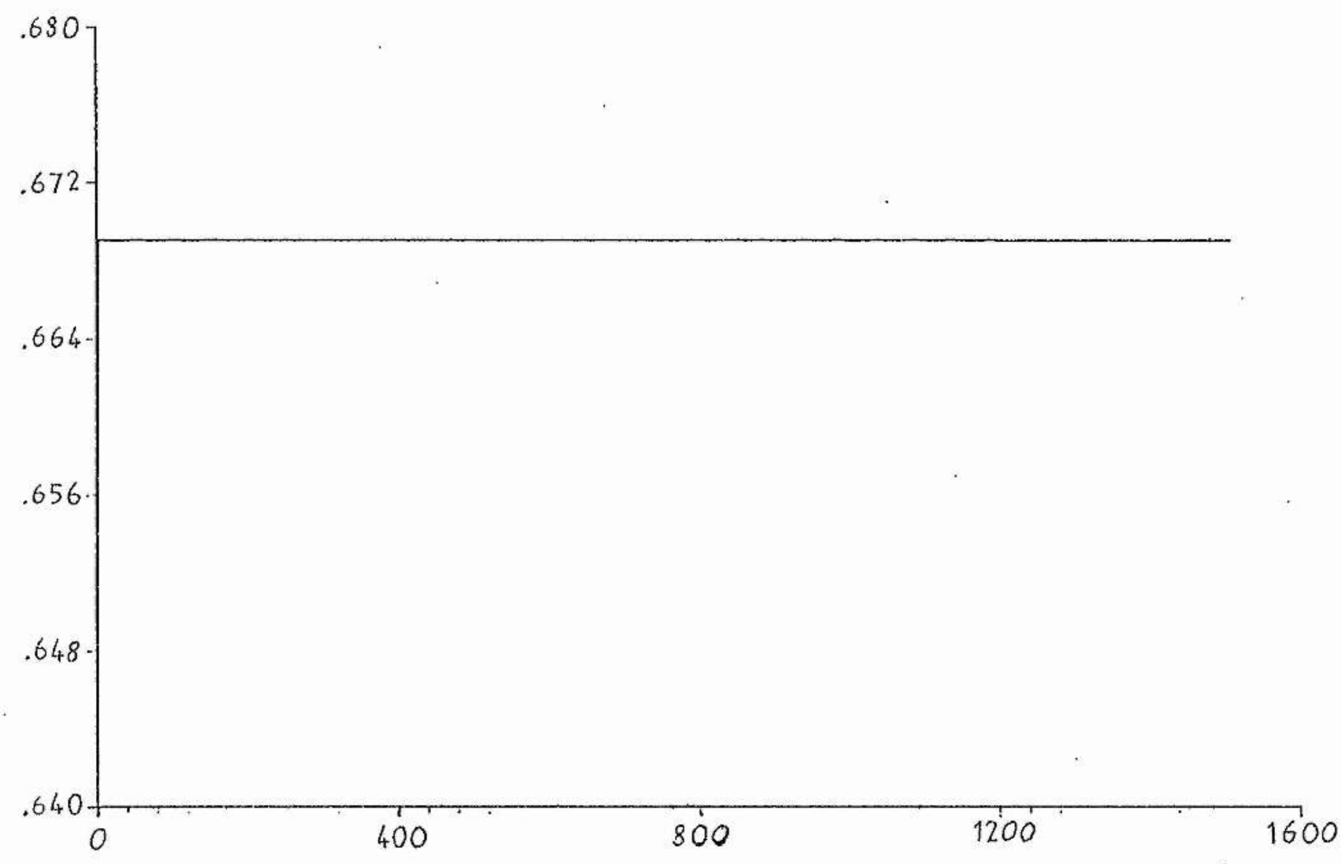
In order to see and understand the behaviour of the orbital parameters, it was much more desirable to illustrate them graphically. For this purpose another program was written, again working from the original results stored on magnetic disc.

This program first went through the results, calculating the parameter a at each step, and plotted this against the time t (in computational time-units). A second graph was obtained in the same way, of e against t. No attempt was made to plot ϖ against t; instead, a graph was obtained of ψ_2 against ψ_1. This is effectively a polar curve, with radius equal to e and with angle equal to ϖ. The loss of information on the rate of change of ϖ with respect to time is offset by the greater understanding obtained from the polar curve; it is basically circular, but in most cases carries periodical fluctuations which may extend to definite "epicycles", during which the movement of the perihelion actually changes direction (see, for example, page 175). The long-term behaviour of e, which is not always immediately obvious from the graph of e against t, can be deduced from the polar diagram of ψ_2 against ψ_1, and the close relationships between the changes in e and in $\tilde{\omega}$ are clearly seen.

The program was originally designed also to calculate and plot σ, Schubart's variable which he uses to warn of close approaches. However, it was found that σ changed monotonically in all the cases investigated (except for (153) Hilda), and so could not usefully be plotted against time.

The program continued by calculating i at each step, and plotting it against time, and finally by plotting χ_2 against χ_1 in the same sort of polar curve as for ψ_2 against ψ_1. (In both cases, care was taken to mark which end of the curve corresponded to the beginning of the integration.) This time the radius of the curve is the sine of the inclination (which, for the fairly small angles in question, reflects qualitatively the behaviour of the inclination itself); and the angle is Ω. Once again, the related changes in the inclination and in the position of the node can be seen very easily.

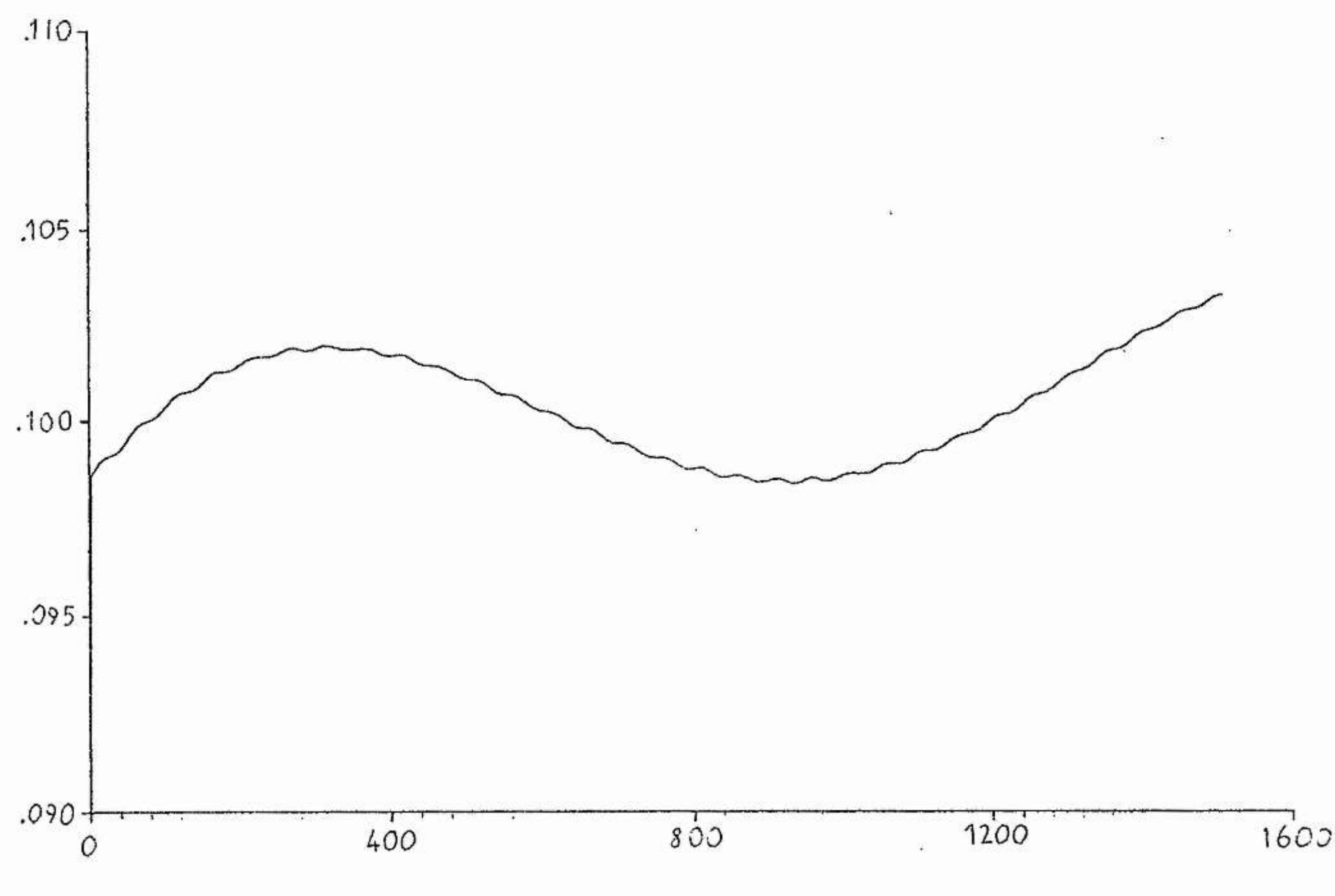
The results of the integrations are presented, in this thesis, in the form of graphs obtained from this program. (The computer listings of the orbital parameters are not reproduced.) The semi-major axes are shown in the units used in the integration, that is, as fractions of the semi-major axis of Jupiter; the time-scales are also in the units used in the integration, with one unit approximately equal to 1.9 years; the inclinations are given in degrees.



(87) Sylvia

$$\frac{p+q}{q} = \frac{9}{5}$$

a against t

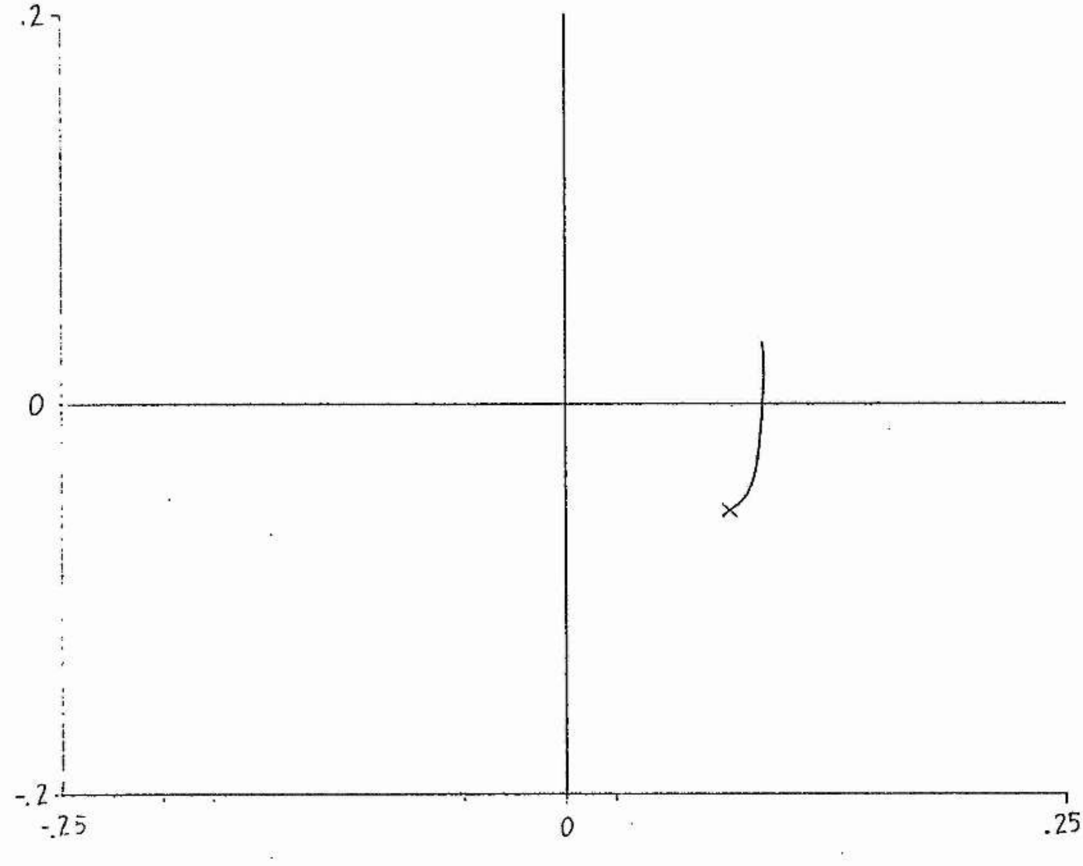


(37) Sylvia

$$\frac{p+q}{q} = \frac{9}{5}$$

e against t

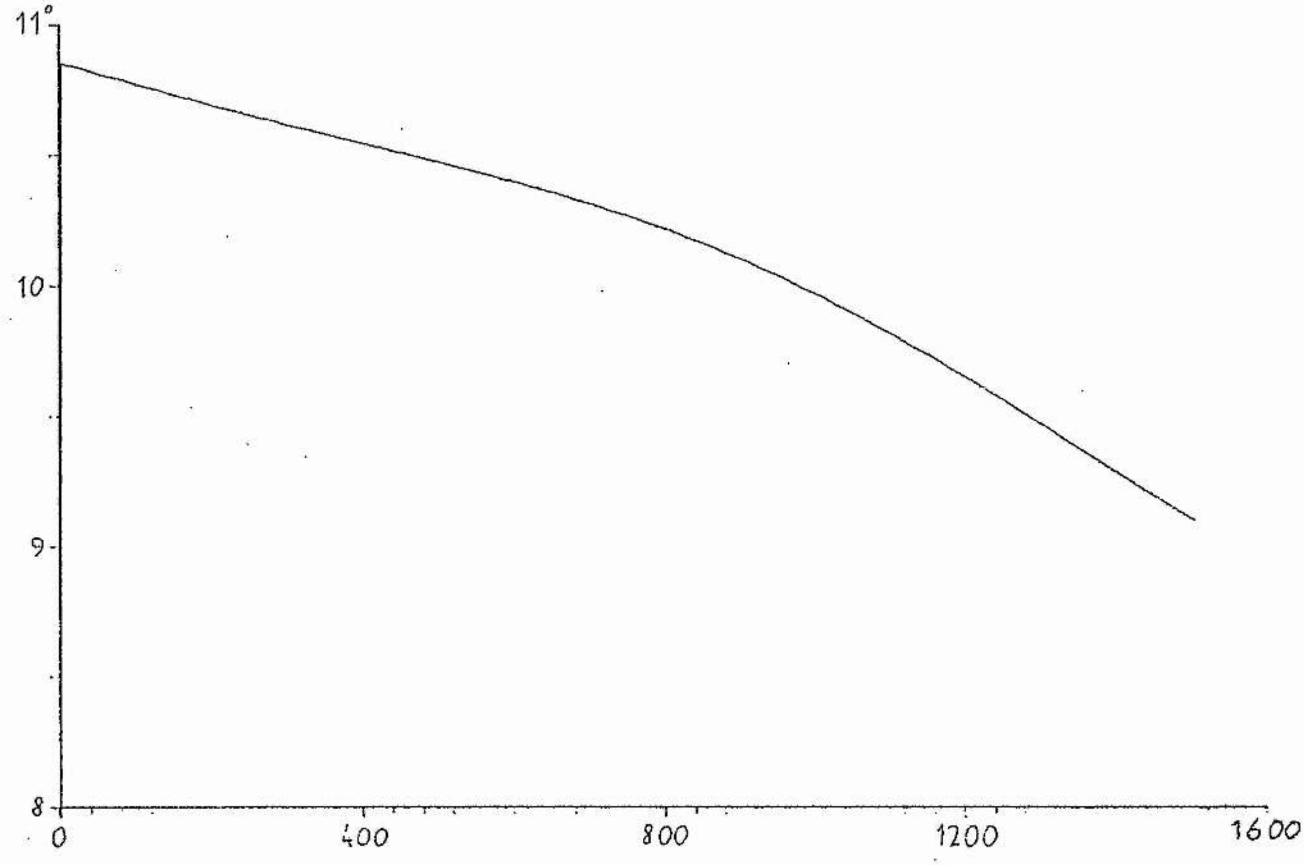
curve begins at X



(87) Sylvia

$$\frac{p+q}{q} = \frac{9}{5}$$

ψ_2 against ψ_1

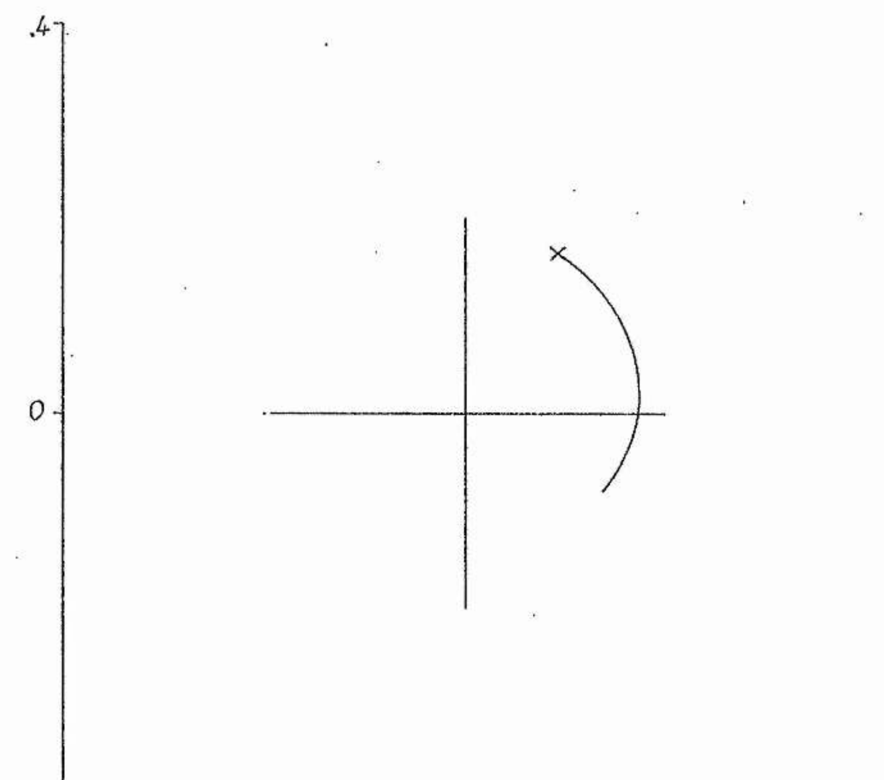


(87) Sylvia

$$\frac{p+q}{q} = \frac{9}{5}$$

i against t

curve begins at x



(87) Sylvia

$$\frac{p+q}{q} = \frac{9}{5}$$

χ_2 against χ_1

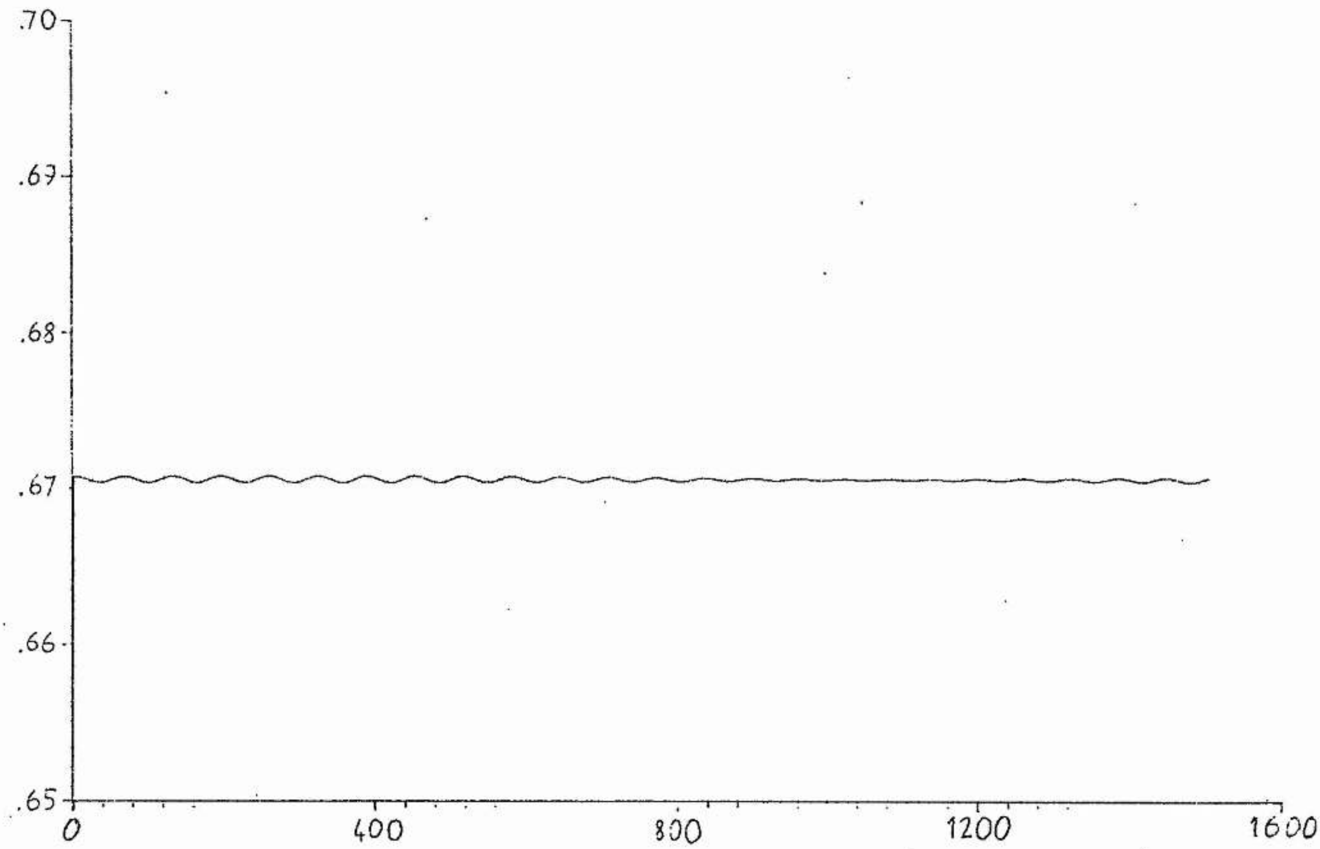
This integration of the orbit of (87) Sylvia covers a period of 1507 time-units, or just over 2800 years. Over this time, the semi-major axis remains virtually constant at 0.6691 units (the unit is the semi-major axis of Jupiter's orbit) - about 3.48 A.U. The listing of the integration results indicates very small oscillations in this value - too small to appear on the graph at the scale used here. They have an amplitude of about 0.00002 units (about 0.003% of the average value) and a period of about 50 time-units, or almost a century.

The eccentricity also has a small-scale oscillation, with the same period as that of the semi-major axis, but opposite in phase. It has an amplitude of about 0.0002, or around 0.2% of the average value of the eccentricity. This small-scale oscillation is superimposed on a slower, larger-amplitude variation of between 0.098 and 0.103, over a period which would appear to be around 1500 time-units, or nearly 3000 years.

The curve of ψ_2 against ψ_1 can be regarded as an approximation to a circle, with a slow variation in radius (that is, in eccentricity), as well as the small rapid oscillation described above. The longitude of perihelion increases steadily from 326 degrees through zero to 18 degrees over the period of the integration (a rate of about 0.018 degrees/year); it passes through zero at t = 1022 units. One cusp can be seen on this curve, corresponding to a maximum in the eccentricity; there is some indication of another, further round by about 50 degrees in longitude of perihelion. There is not really enough of this curve for a circle to be fitted to it with any accuracy, but a circle of radius 0.09, centred at (0.01, 0), fits tolerably well.

The graph of the inclination shows only that it decreases from 10.8 degrees to 9.4 degrees over the period of the integration. The nature of this becomes clearer from examination of the curve X_2 against X_1. This is a very good approximation to a circle of radius about 0.18, corresponding to an inclination of 10 degrees, centred at (0, 0.02), giving an offset of 1.4 degrees inclination. This suggests that the graph of the inclination, if continued, would show a minimum of 8.6 degrees at about 2500 time-units, and then complete what would be approximately a sine-curve in a period of around 6000 time-units.

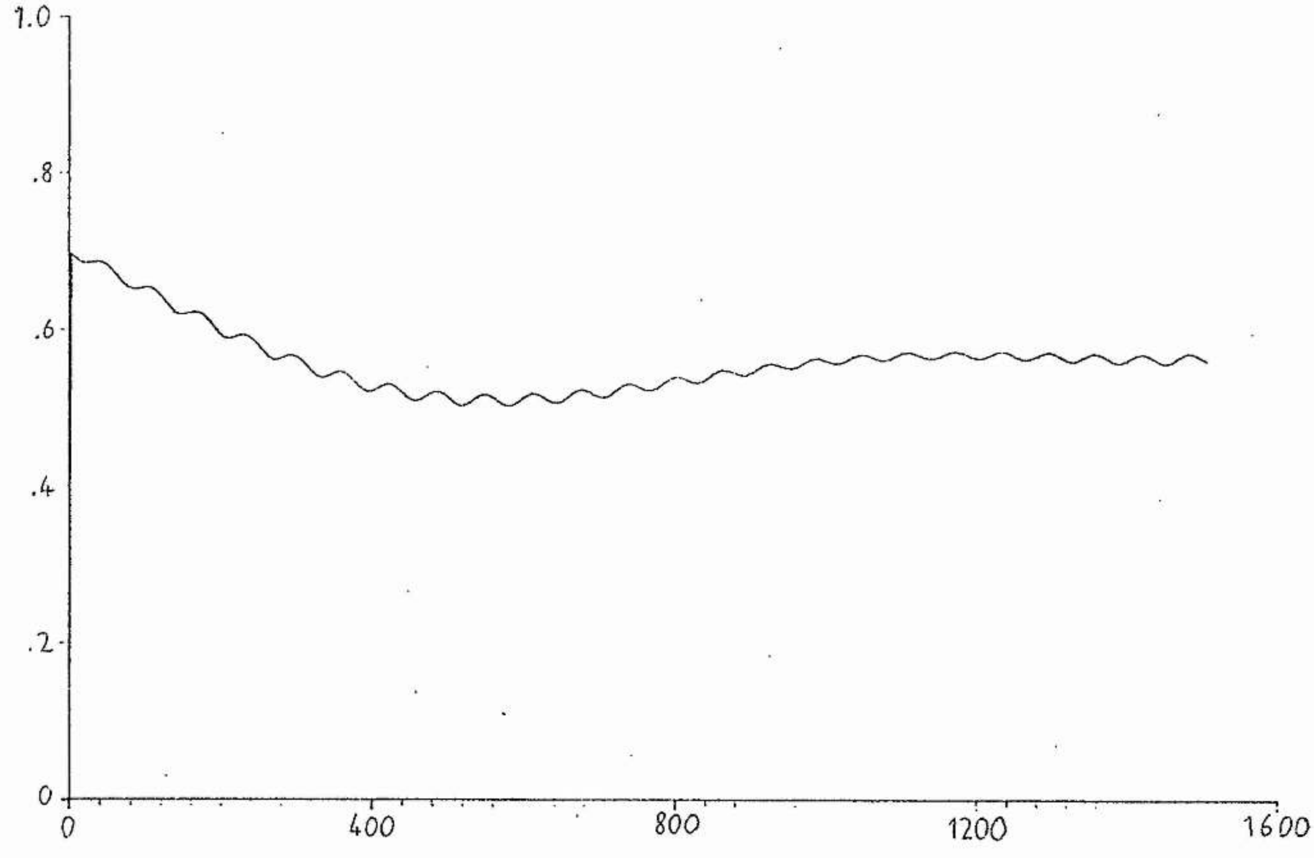
The ascending node retrogresses steadily from 61 degrees to 330 degrees, passing through zero at t = 1030 units (a rate of 0.032 degrees/year). This means that the ascending node is moving almost twice as fast as the perihelion. The critical angle σ was found from the printed output to decrease monotonically, through almost eight revolutions (a rate of about 1 degree/year); however, the minor planet did not approach Jupiter more closely than 0.25 units (1.3 A.U.) throughout the integration.



(107) Camilla

$$\frac{p+q}{q} = \frac{9}{5}$$

a against t

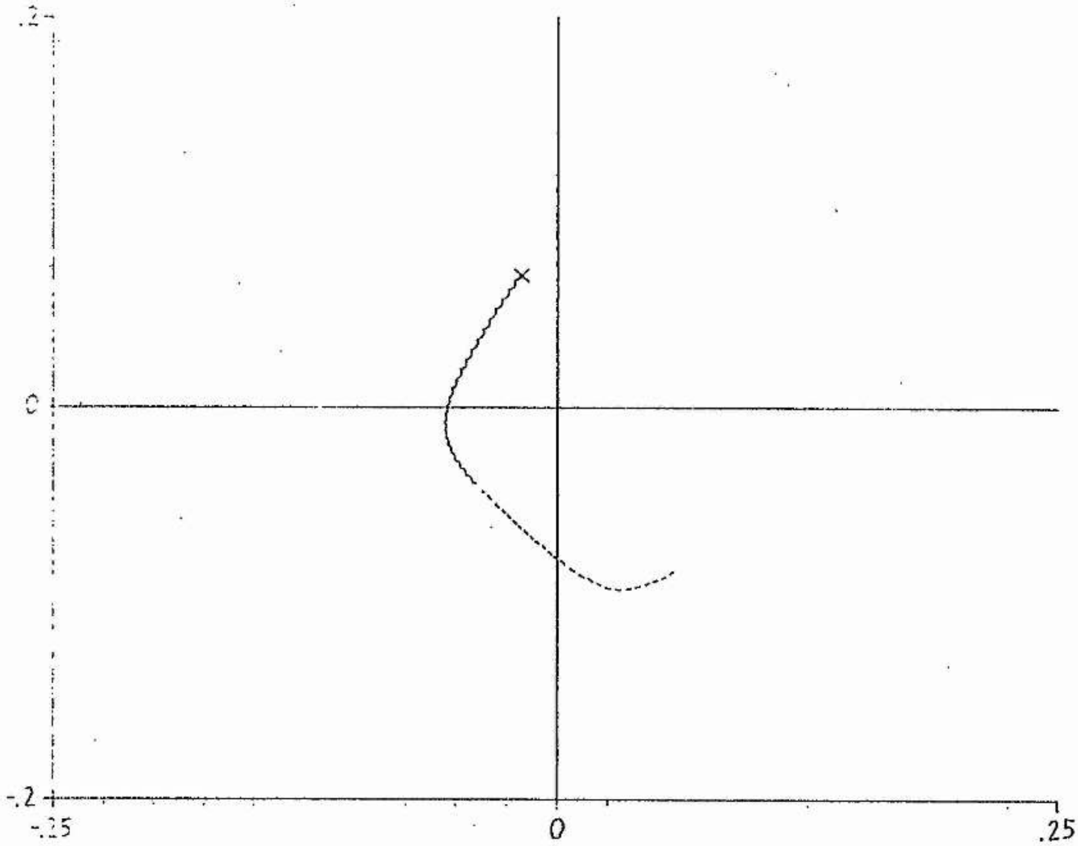


(107) Camilla

$$\frac{p+q}{q} = \frac{9}{5}$$

e against t

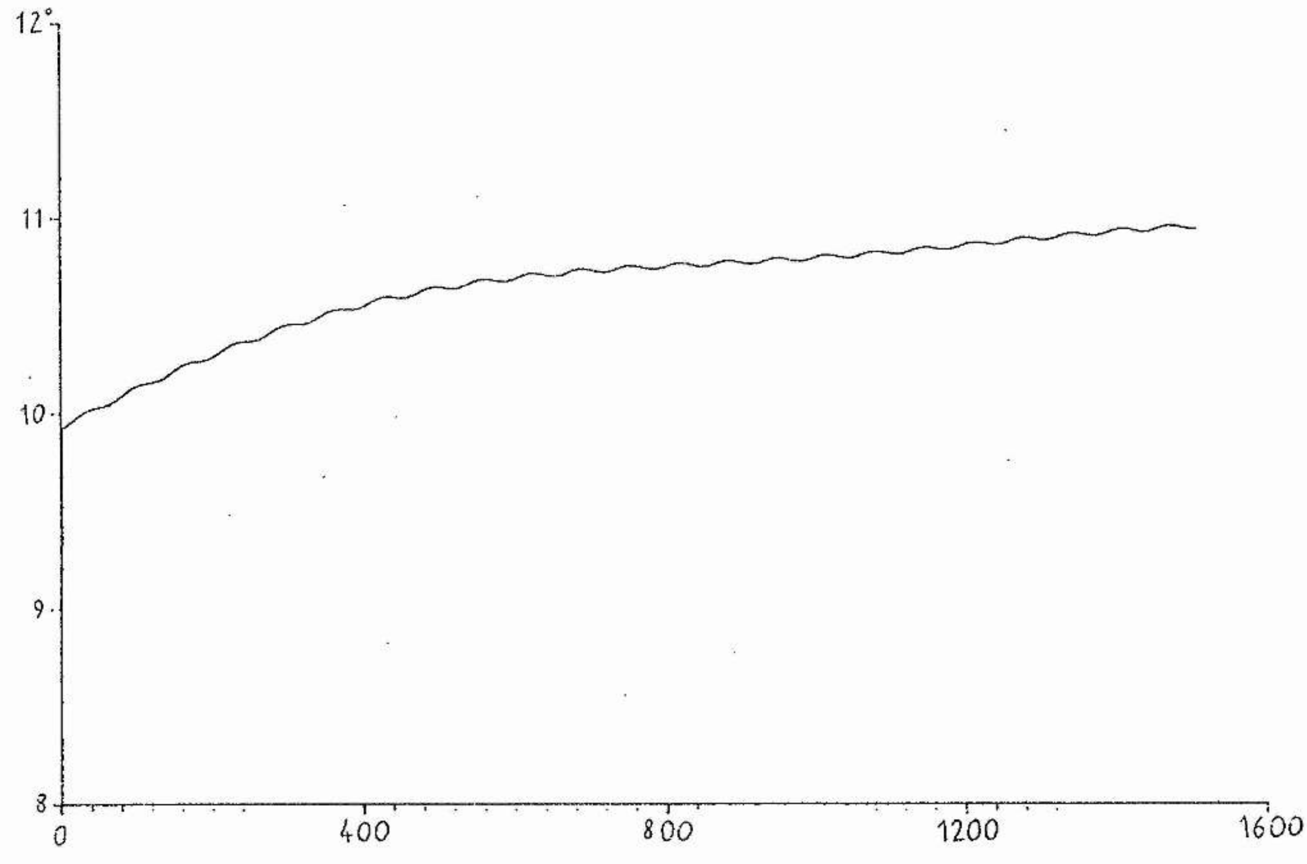
curve begins at X



(107) Camilla

$$\frac{p+q}{q} = \frac{9}{5}$$

ψ_2 against ψ_1

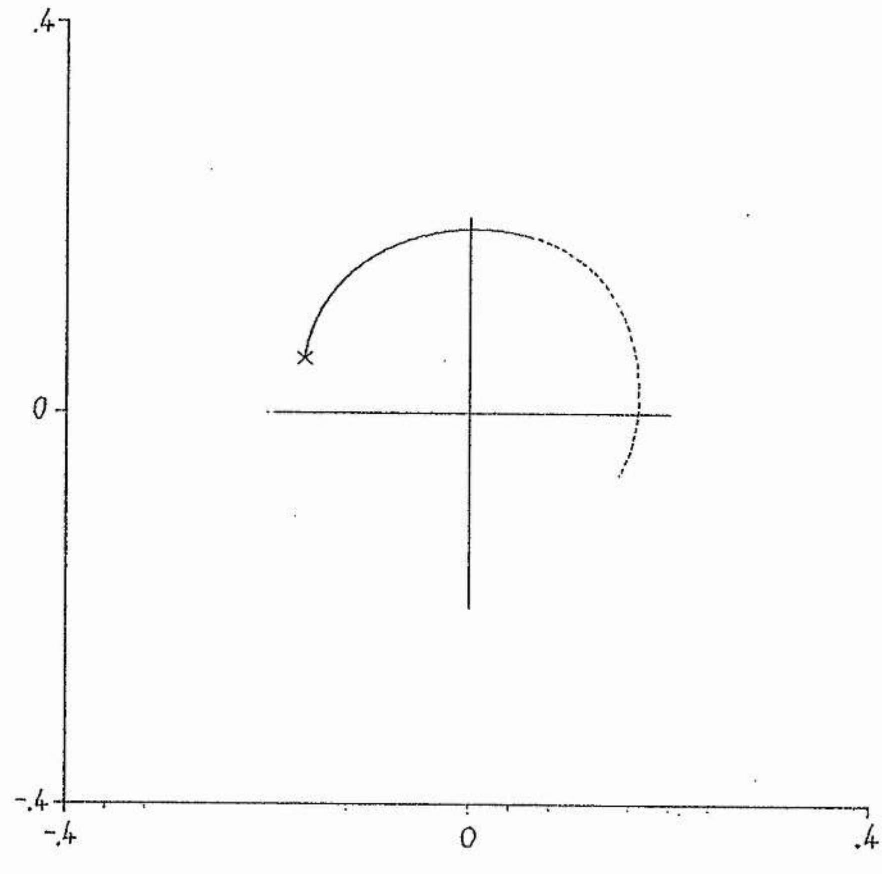


(107) Camilla

$$\frac{p+q}{q} = \frac{9}{5}$$

i against t

curve begins at x



(107) Camilla

$$\frac{p+q}{q} = \frac{9}{5}$$

χ_2 against χ_1

The integration of the orbit of (107) Camilla was carried out in two stages, covering a total interval of 3010 time-units. Unfortunately it was not possible to plot the entire results on one set of graphs, by computer, as the two stages of the integration were separated by other calculations, and the original results were destroyed before the second set were created. The graphs reproduced here are those for the first half of the integration, but the two polar diagrams have been extended with a dotted line to show the subsequent course of the curve during the second half.

The semi-major axis is essentially constant at 0.67 units (3.49 A.U.), but carries a small oscillation with a period of about 64 time-units. The amplitude of this oscillation also varies with time, varying from around 0.03% of the average value down to almost zero; the period of this variation appears to be roughly 1500 time-units.

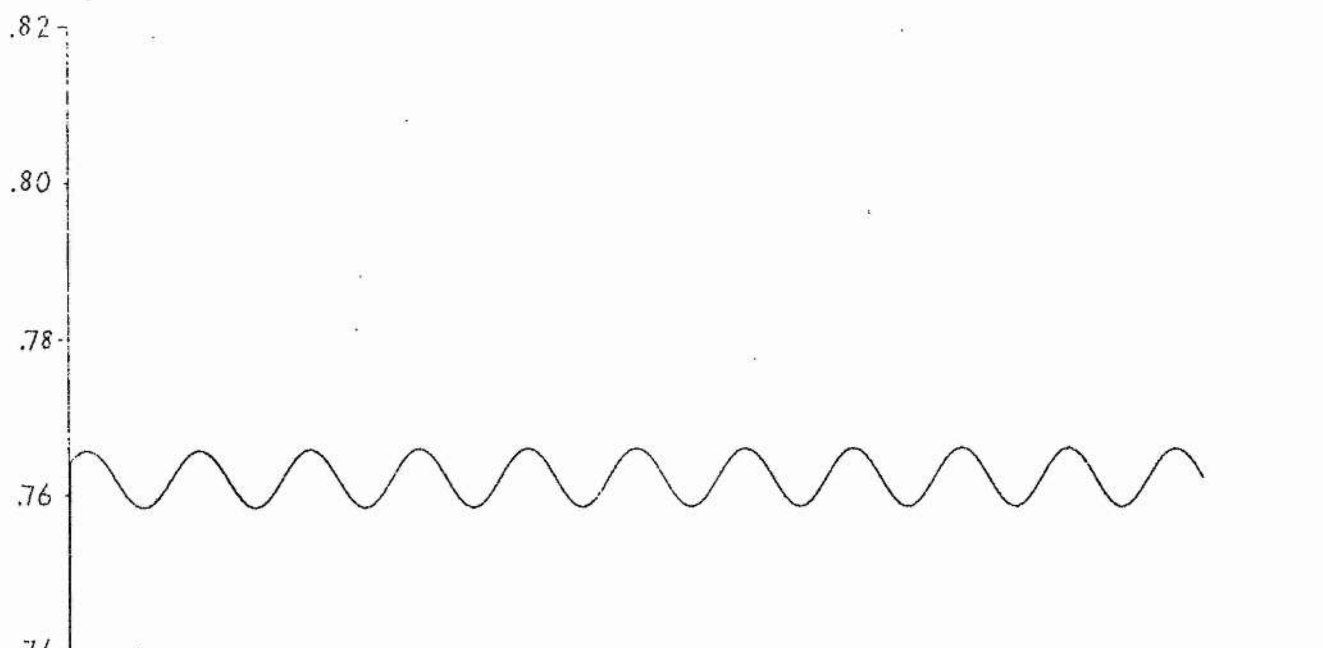
The eccentricity carries a similar small oscillation, of the same period but opposite phase; this however is of approximately constant amplitude, about 0.15% of the average value. This is superimposed on a much larger oscillation, causing the eccentricity to vary between 0.05 and over 0.10. This oscillation appears to be connected with the "amplitude modulation" of the semi-major axis: the extremes of the oscillation in eccentricity, where it is changing only slowly, correspond to the smallest fluctuations in the semi-major axis; the points where the eccentricity is changing most rapidly correspond to the largest fluctuations in the semi-major axis.

The curve of ψ_2 against ψ_1 approximates to a circle, with cusps at intervals of around 75 degrees - that is, with the maximum values of the eccentricity occurring at intervals of 75 degrees in the longitude of the perihelion. Over the entire period of the integration, the perihelion advances steadily from 105 degrees to 304 degrees (a rate of 0.035 degrees/year); it passes through 180 degrees at t = 880, and through 270 degrees at t = 2110 time-units. A circle can be drawn with centre (0.04, 0) and radius 0.088, which fits the ψ-curve reasonably well.

The inclination, like the semi-major axis and the eccentricity, carries a small oscillation, of amplitude about 0.02 degrees; here the period is approximately 68 time-units, which is close to, but different from, the period of the oscillations in the first two parameters. At the end of the first half of this integration, the inclination reaches a maximum of almost 11 degrees, falling to just over 9 degrees at the beginning and end.

The curve of χ_2 against χ_1 is almost exactly a circle of radius 0.17, centred on (0, 0.02); this corresponds to a constant inclination of 9.8 degrees with an offset of 1.2 degrees. Over the full integration, the ascending node retrogresses from 161 degrees to 337 degrees, passing through 90 degrees at t = 1187, and through zero at t = 2670 time-units. Thus the ascending node is moving at around 0.032 degrees/year - a little more slowly than the perihelion.

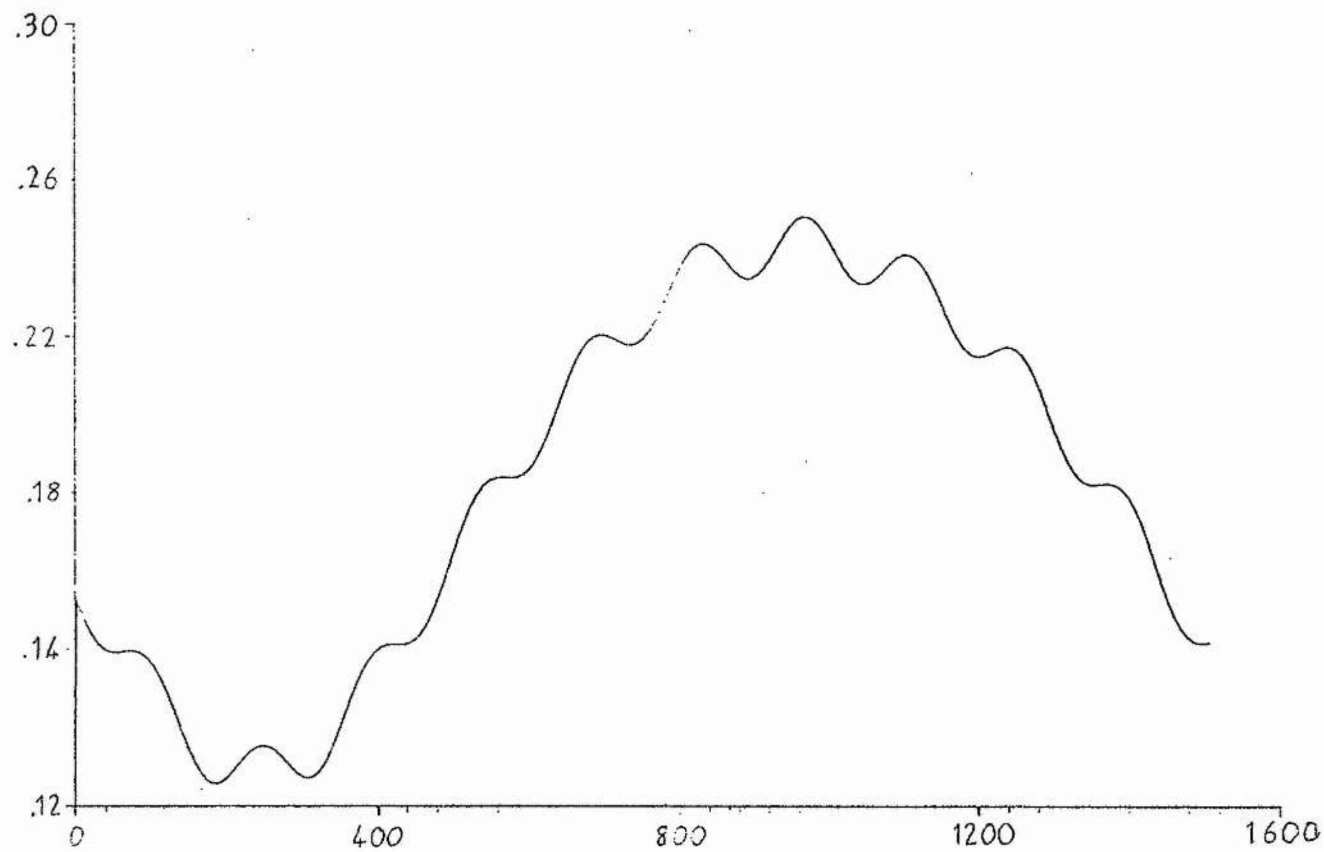
The critical angle σ is found to decrease monotonically throughout the integration, at about 0.4 degrees/year, but no approach closer than 0.25 units (1.3 A.U.) occurs.



(153) Hilda

$$\frac{p+q}{q} = \frac{3}{2}$$

a against t

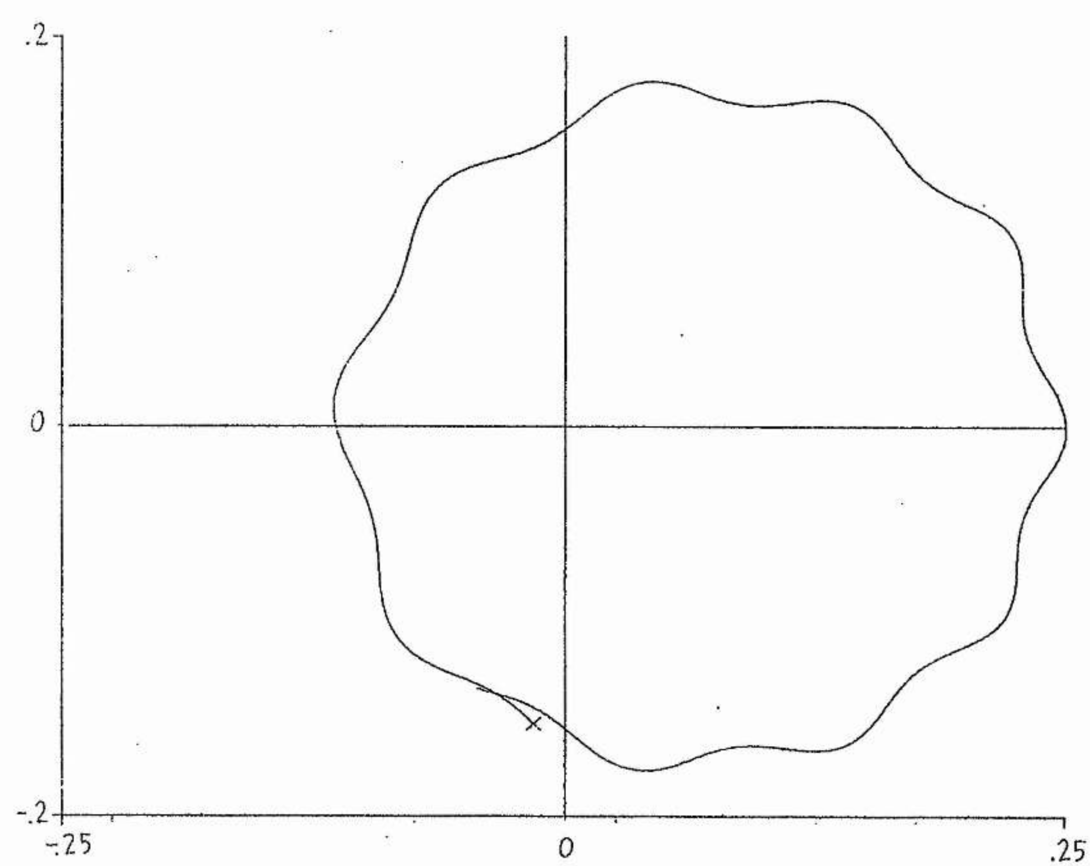


(153) Hilda

$$\frac{p+q}{q} = \frac{3}{2}$$

e against t

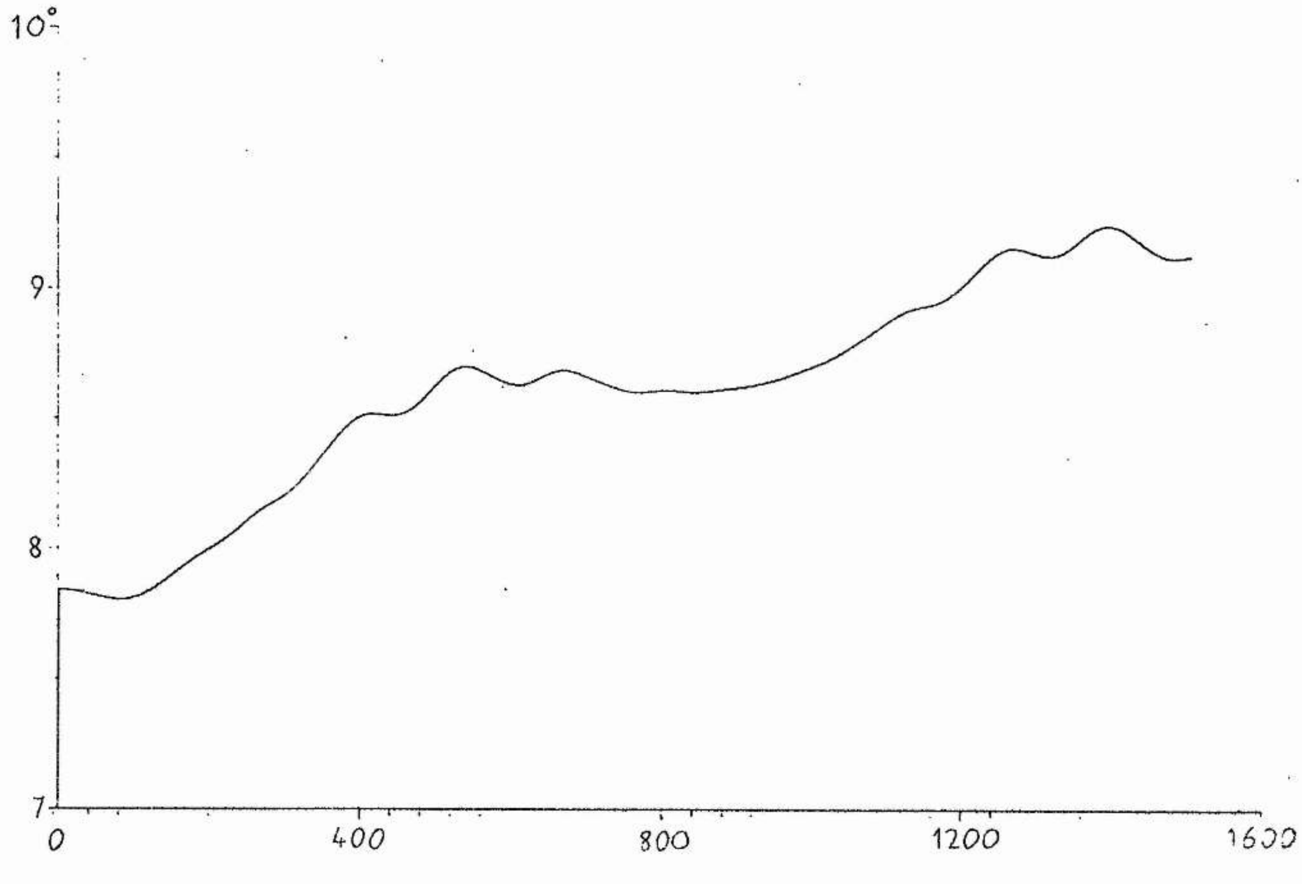
curve begins at X



(153) Hilda

$$\frac{p+q}{q} = \frac{3}{2}$$

ψ_2 against ψ_1

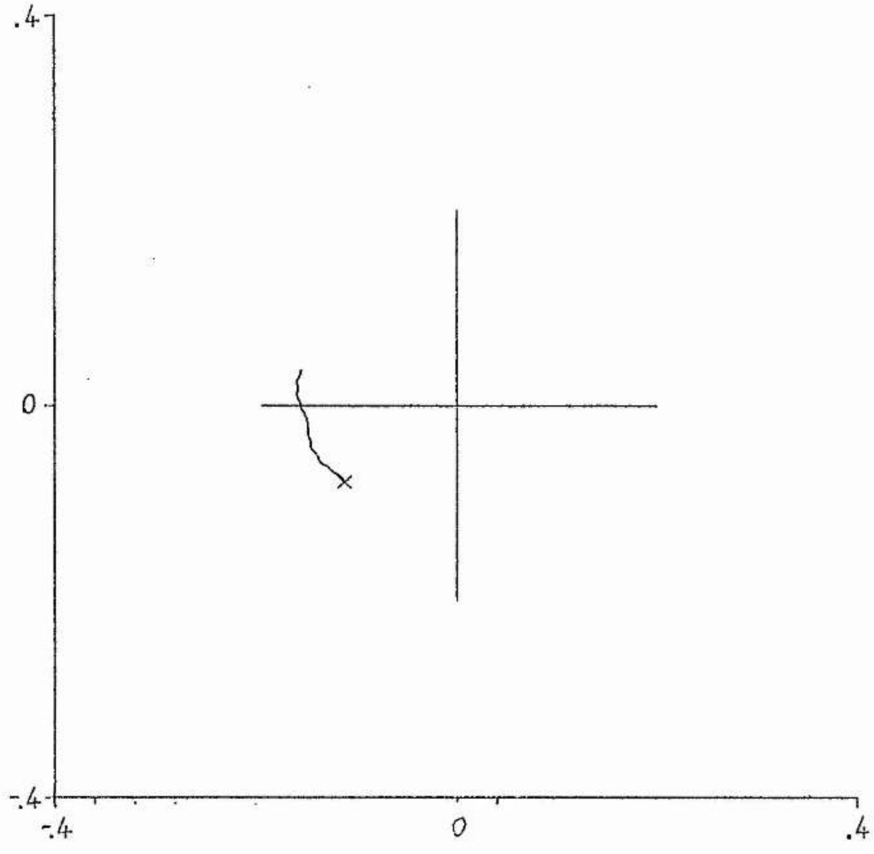


(153) Hilda

$$\frac{p+q}{q} = \frac{3}{2}$$

i against t

curve begins at X



(153) Hild3

$$\frac{p+q}{q} = \frac{3}{2}$$

X_2 against X_1.

The integration of the orbit of (153) Hilda, taking into account the inclination, was carried out over a period of 1507 time-units. During this period, the semi-major axis is seen to oscillate around 0.761 units (3.96 A.U.) with an amplitude of about 0.5%, and a period of 18.5 time-units. The amplitude remains almost constant throughout the period of the integration.

The eccentricity shows a similar small oscillation, with the same period but opposite phase; here the amplitude is about 4% of the average value. This is superimposed on a larger variation, with a period of about 1500 time-units; this causes the eccentricity to vary between 0.11 and 0.24.

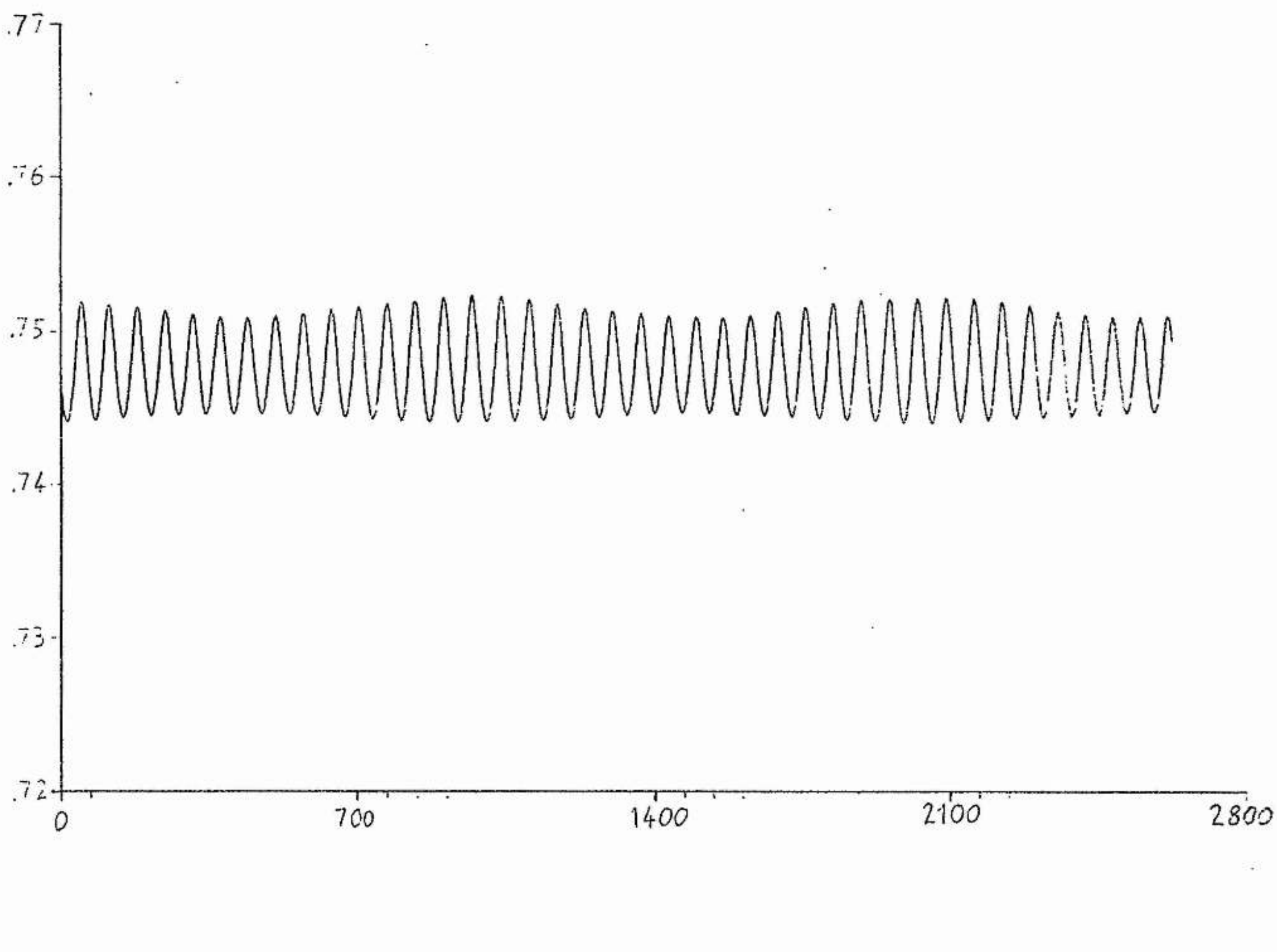
The curve of ψ_2 against ψ_1 is basically a circle with small variations in radius (i.e. eccentricity); just over ten cusps occur in one revolution, showing that the small oscillations in eccentricity have a period just under one-tenth of the time taken for one circuit of the perihelion. A circle may be drawn to fit the curve, having centre at (0.068, 0) and radius of 0.172. During the period of the integration, the perihelion retrogresses by a little over one revolution, moving at an average speed of 0.12 degrees/year; it crosses the four axes of the plot at times of 230, 480, 965 and 1450 units.

The curve of the inclination is complex, having a small oscillation which only appears at the peaks of a larger one. The periods of these two variations are about 147 and 850 time-units. The amplitude of the smaller oscillation is about 0.1 degrees; the larger variation moves the inclination from 7.8 to 9.2 degrees in the course of the integration.

The curve of χ_2 against χ_1 is apparently part of a circle, with variations in radius caused by the variations in the inclination; however, there is not sufficient of the curve to estimate the centre or radius of the circle. The ascending node regresses from 214 to 167 degrees, passing through 180 degrees at 1150 time-units. It moves much more slowly than the perihelion, at a rate of only 0.02 degrees/year.

At the beginning of the integration, the critical angle σ is found to oscillate between about -45 and +11 degrees, with a period of around 145 time-units. This verifies that no "close approaches" between the minor planet and Jupiter occur. Unfortunately, the data were not analysed to show the behaviour of σ in detail before they were destroyed, so no direct comparison can be made with the resuls given by Schubart (1978); sample calculations on the figures which are available suggest reasonably good agreement.

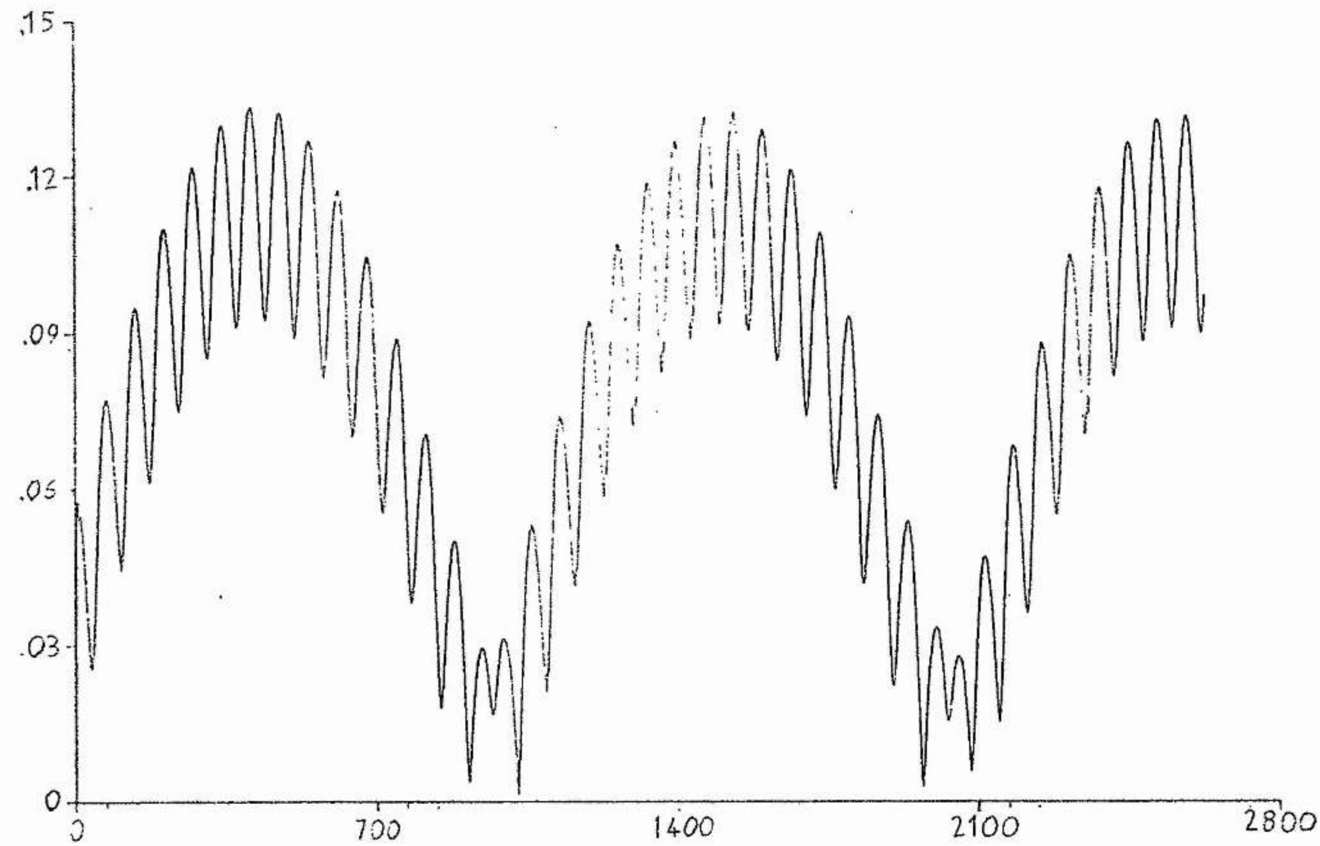
These graphs may be compared with diagram 8, where some similar graphs are shown, at a reduced scale, for the integration of the same orbit without including the inclination.



(334) Chicago

$$\frac{p+q}{q} = \frac{3}{2}$$

a against t

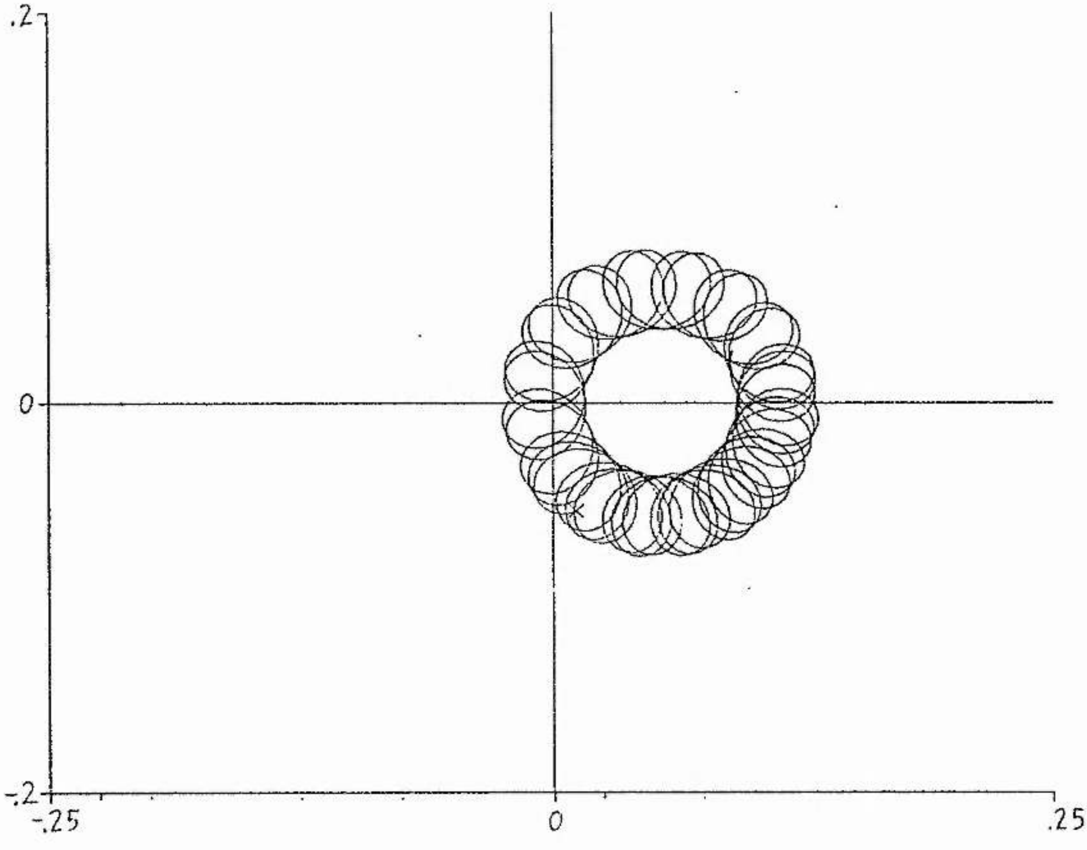


(334) Chicago

$$\frac{p+q}{q} = \frac{3}{2}$$

e against t

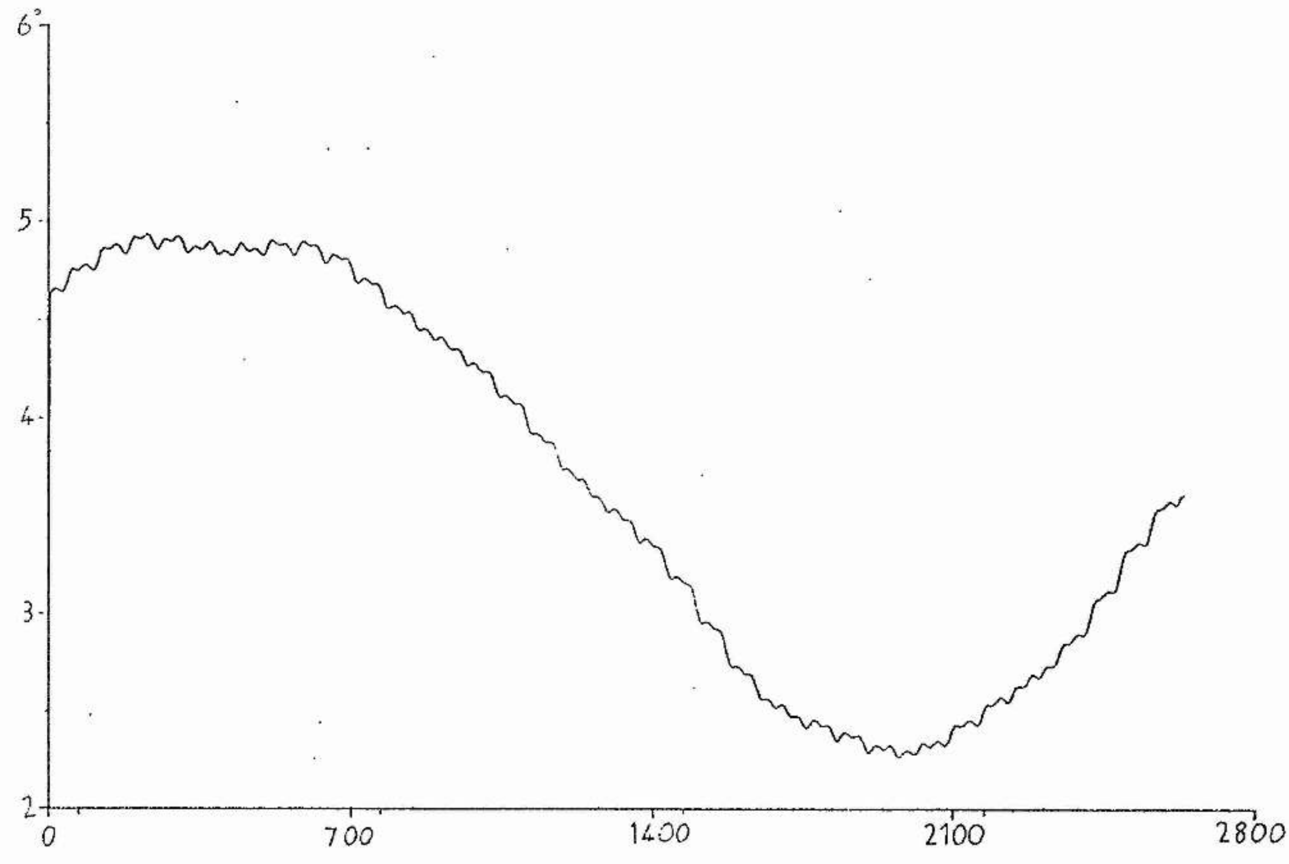
curve begins at X



(334) Chicago

$$\frac{p+q}{q} = \frac{3}{2}$$

ψ_2 against ψ_1

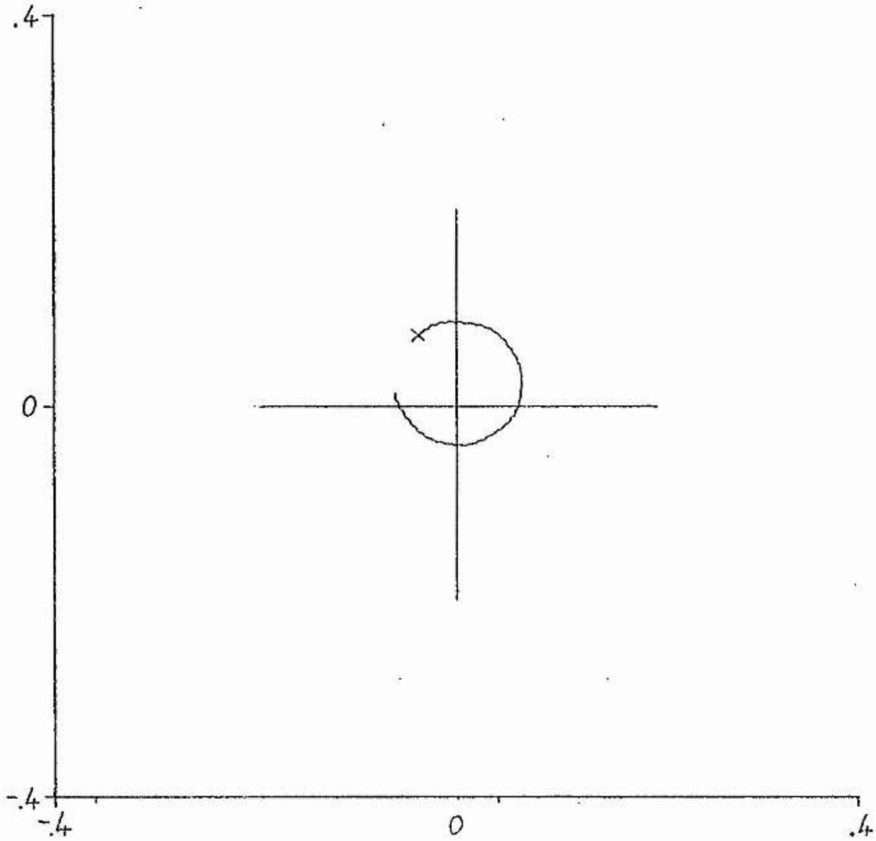


(334) Chicago

$$\frac{p+q}{q} = \frac{3}{2}$$

i against t

curve begins at X



(334) Chicago

$$\frac{p+q}{q} = \frac{3}{2}$$

χ_2 against χ_1

The integration of the orbit of (334) Chicago was carried out over an interval of 2630 time-units (about 5300 years). It was possible to carry this further than the integrations for most of the other planets, in the space of one night's computing, since the ratio of mean motions was taken as 3/2; this meant that p and q were smaller, and that many of the processes in the calculations took less time.

The semi-major axis remains near 0.748 units (3.89 A.U.), oscillating with a period of about 67 units; the amplitude of these oscillations varies from 0.006 to 0.008 units (around 1% of the average value), this variation having a period of about 1080 time-units.

The eccentricity never exceeds 0.14; it has regular oscillations of amplitude about 0.04, which is considerably larger in proportion than those in the semi-major axis; they have the same period as those of the semi-major axis, but opposite phase. These are superimposed on a larger, slower oscillation with amplitude about 0.1 and period 1080 time-units - the same as the period of the "amplitude modulation" in the semi-major axis. The combined effect of these two oscillations is to bring the eccentricity down to zero as they both approach their minima; it would in fact be carried below zero if it were not that it is calculated from

$$e = \sqrt{(\psi_1^{\,2} + \psi_2^{\,2})}$$

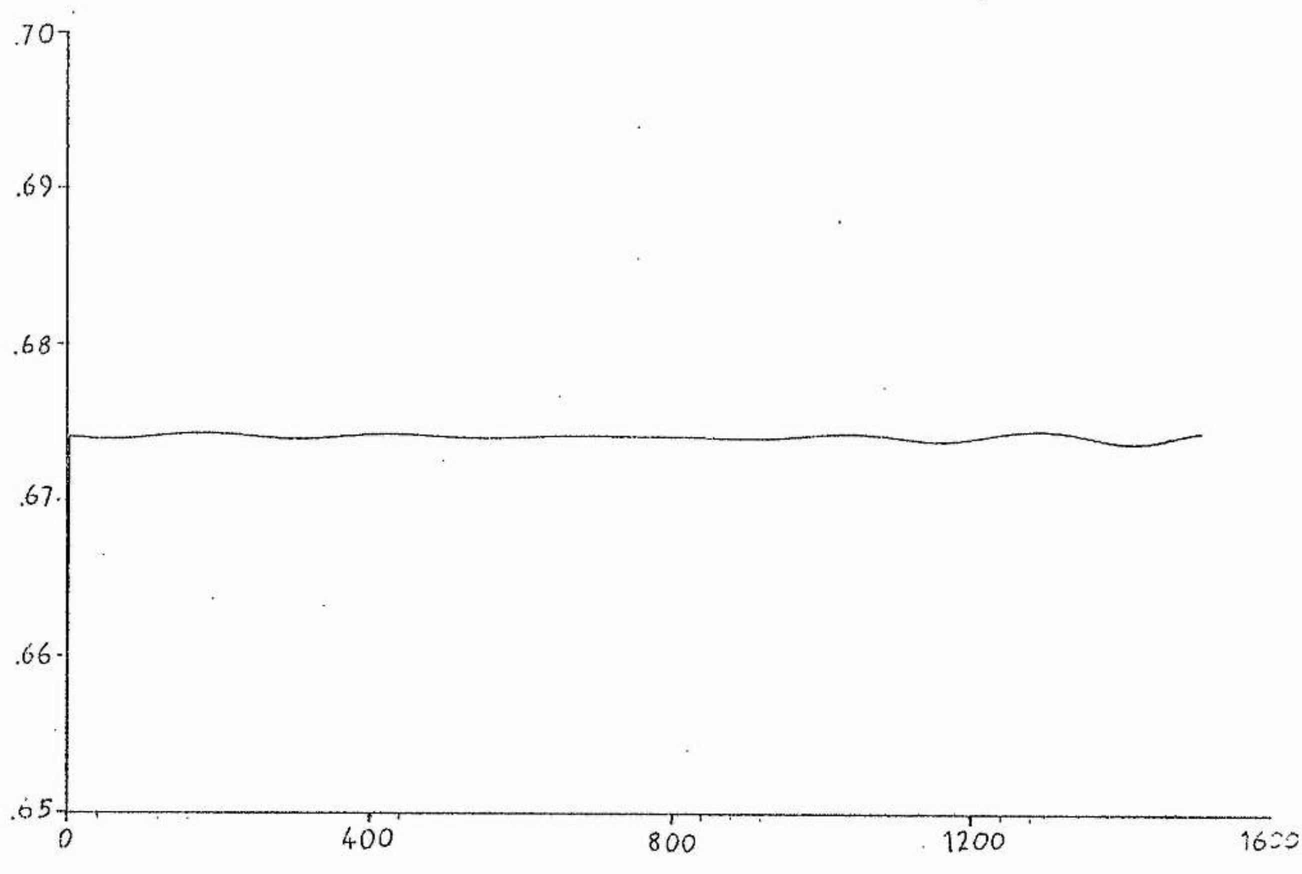
and is always assumed to be positive. This leads to the curve being partially "reflected" from the horizontal axis.

The complex behaviour of the eccentricity at its minima is further elucidated by examining the $\psi_1 - \psi_2$ curve. This shows that the perihelion of the orbit advances slowly, but with frequent retrogressions, the forwards and backwards movements corresponding to the rapid oscillations in the eccentricity. The curve accordingly exhibits retrograde epicycles, which are so large as to bring the curve to the "wrong" side of the axis when the longitude of perihelion is near 180 degrees. The overall shape of the curve is almost exactly circular, with the centre of the circle at (0.055, 0), and an average radius (to the centre of the epicycles) of approximately 0.6.

The graph reproduced here is further complicated by the fact that the perihelion moves very rapidly, and that more than two full revolutions are shown; the period of the epicycles is evidently not commensurable with the overall period, as the pattern is slightly shifted in longitude on each repeat. The perihelion moves from 282 degrees to 19 degrees over the interval of the integration, giving a net advance of 817 degrees, or 0.16 degrees/year; in fact the movement is nearly three times as rapid is this when the epicycles are taken into account, giving a rate of nearly half a degree a year. (This graph corresponds closely with one given by Schubart (1968) for the orbit of (334) Chicago.)

The inclination averages about 3.5 degrees. It shows a small double oscillation, apparently due to two separate small oscillations of slightly different periods. The overall period is around 70 time-units, with an amplitude of about 0.6% of the average value. This double oscillation is superimposed on a larger curve of amplitude 2.5 degrees and period about 3100 time-units. The nature of this larger oscillation is explained by the graph of χ_2 against χ_1, which is an approximation to a circle offset from the centre of the plot. The radius of this circle is 0.06 (corresponding to an inclination of 3.4 degrees) and the centre is (0, 0.02) (inclination 1.2 degrees). The ascending node retrogresses from 118 degrees to 169 degrees (a rate of 0.06 degrees/year). It thus covers rather less than one circle in the time the perihelion takes to advance by more than two. The ascending node passes the values of 90 degrees, zero and 270 degrees at times 345, 1310 and 1945 units.

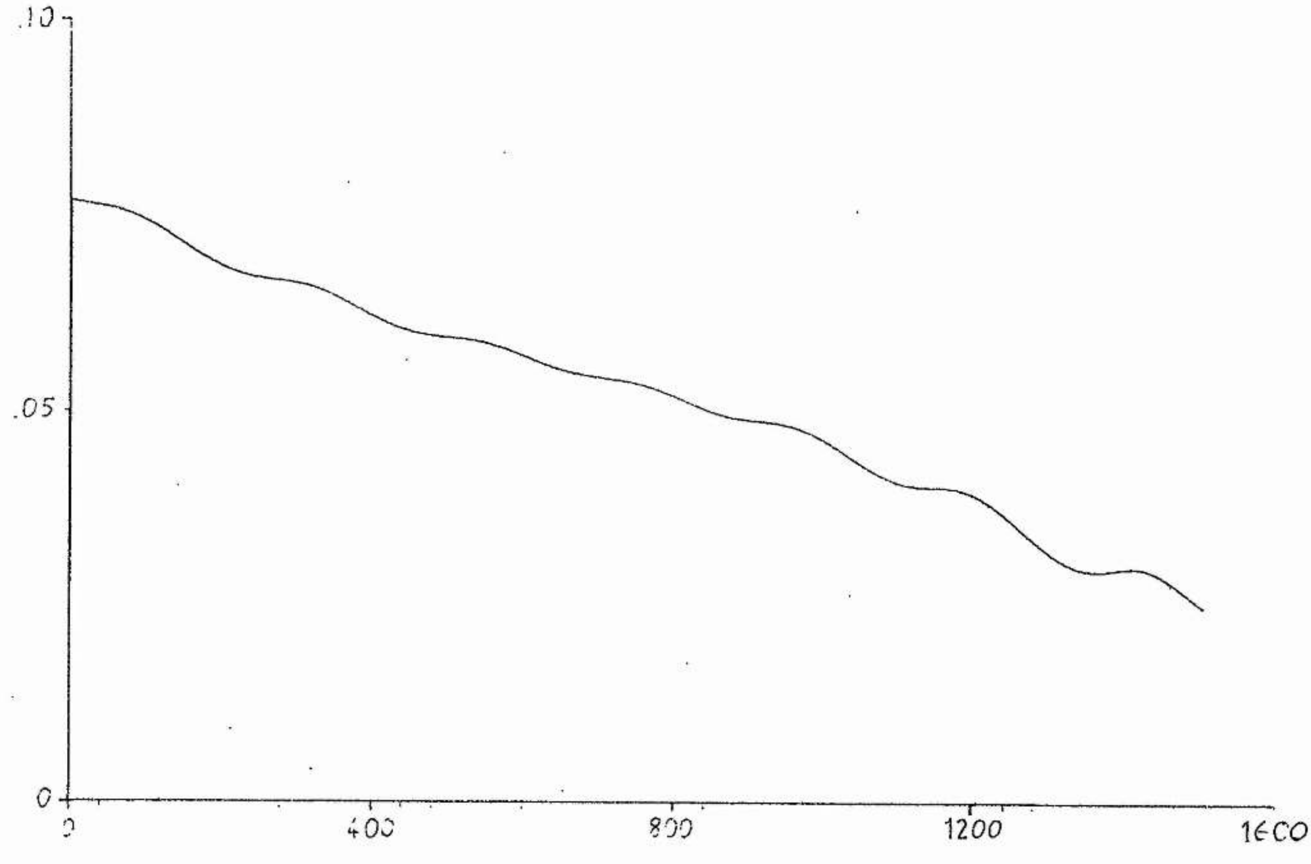
The critical angle σ was found to decrease monotonically, at a rate of nearly 3 degrees/year, over the interval of the integration, but no approach to Jupiter closer than 0.21 units (1.09 A.U.) occurred.



(414) Liriope

$$\frac{p+q}{q} = \frac{9}{5}$$

a against t

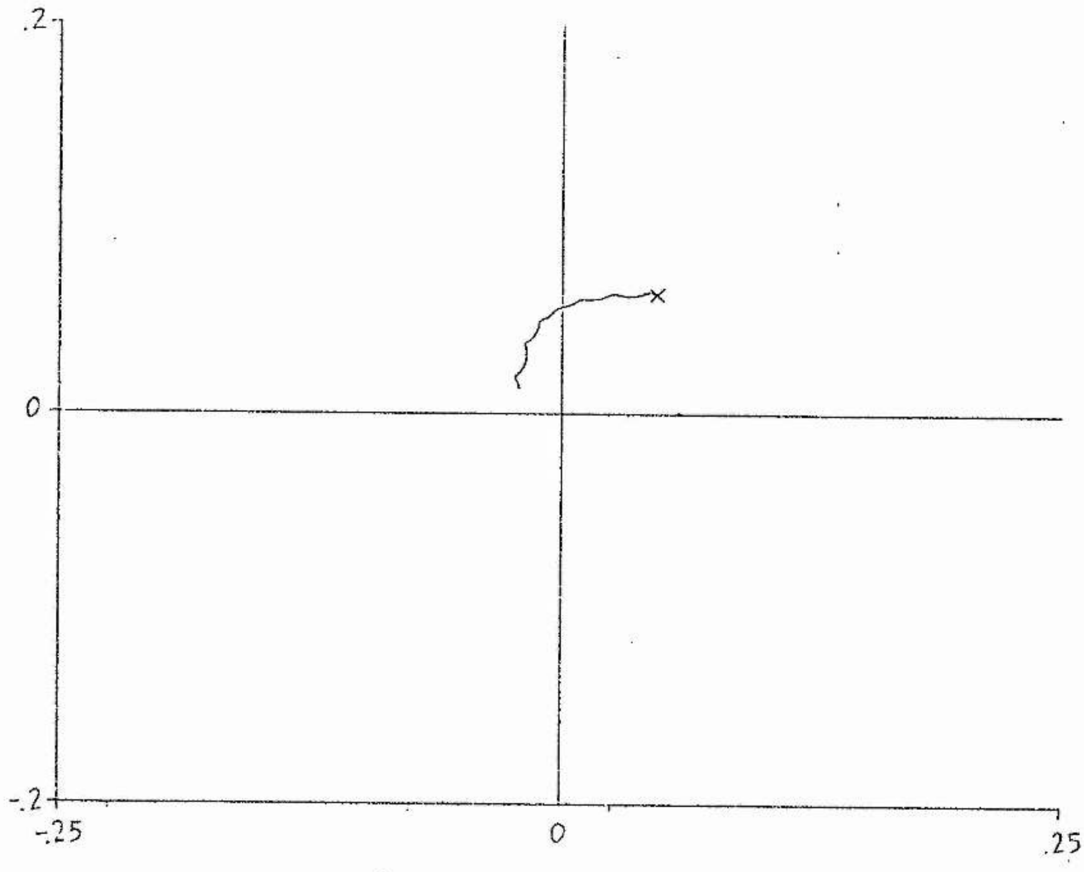


(414) Liriope

$$\frac{p+q}{q} = \frac{9}{5}$$

e against t

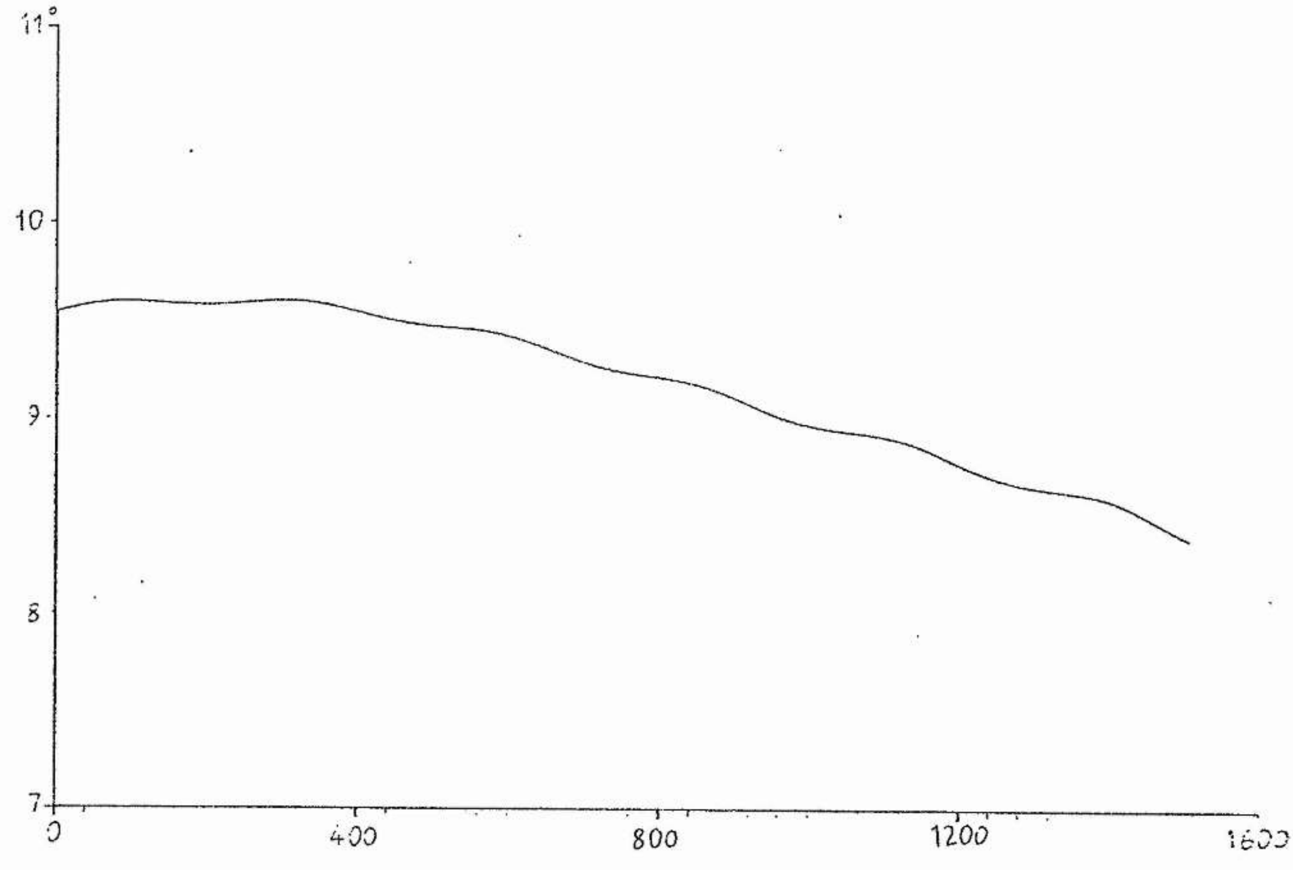
curve begins at X



(414) Liriope

$$\frac{p+q}{q} = \frac{9}{5}$$

ψ_2 against ψ_1

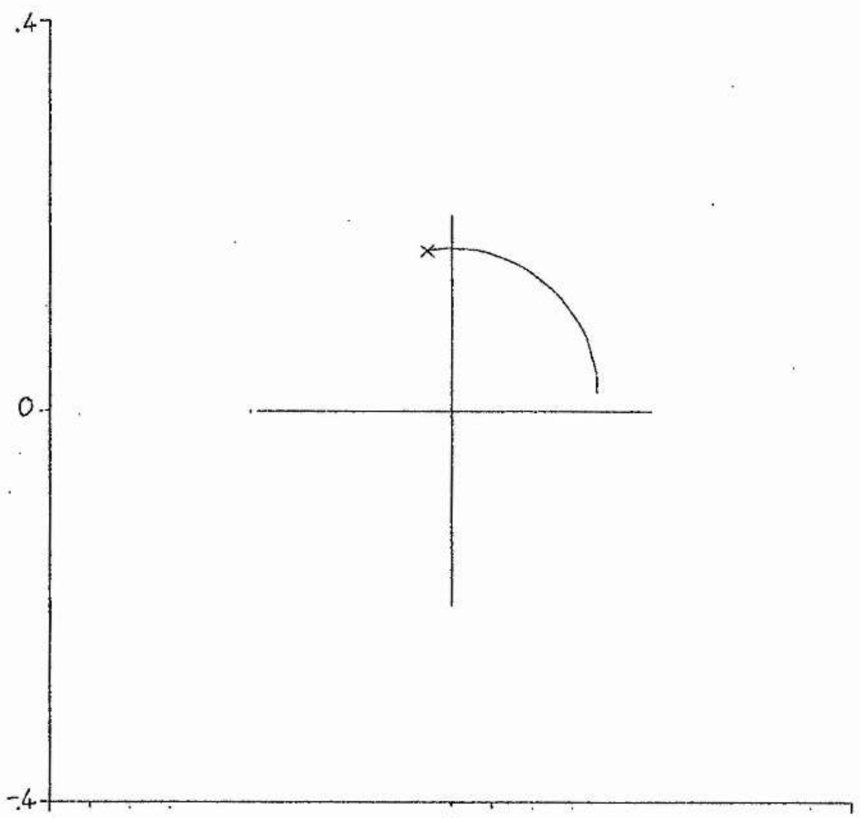


(414) Liriope

$$\frac{p+q}{q} = \frac{9}{5}$$

i against t

curve begins at X



(414) Liriope

$$\frac{p+q}{q} = \frac{9}{5}$$

X_1 against X_1

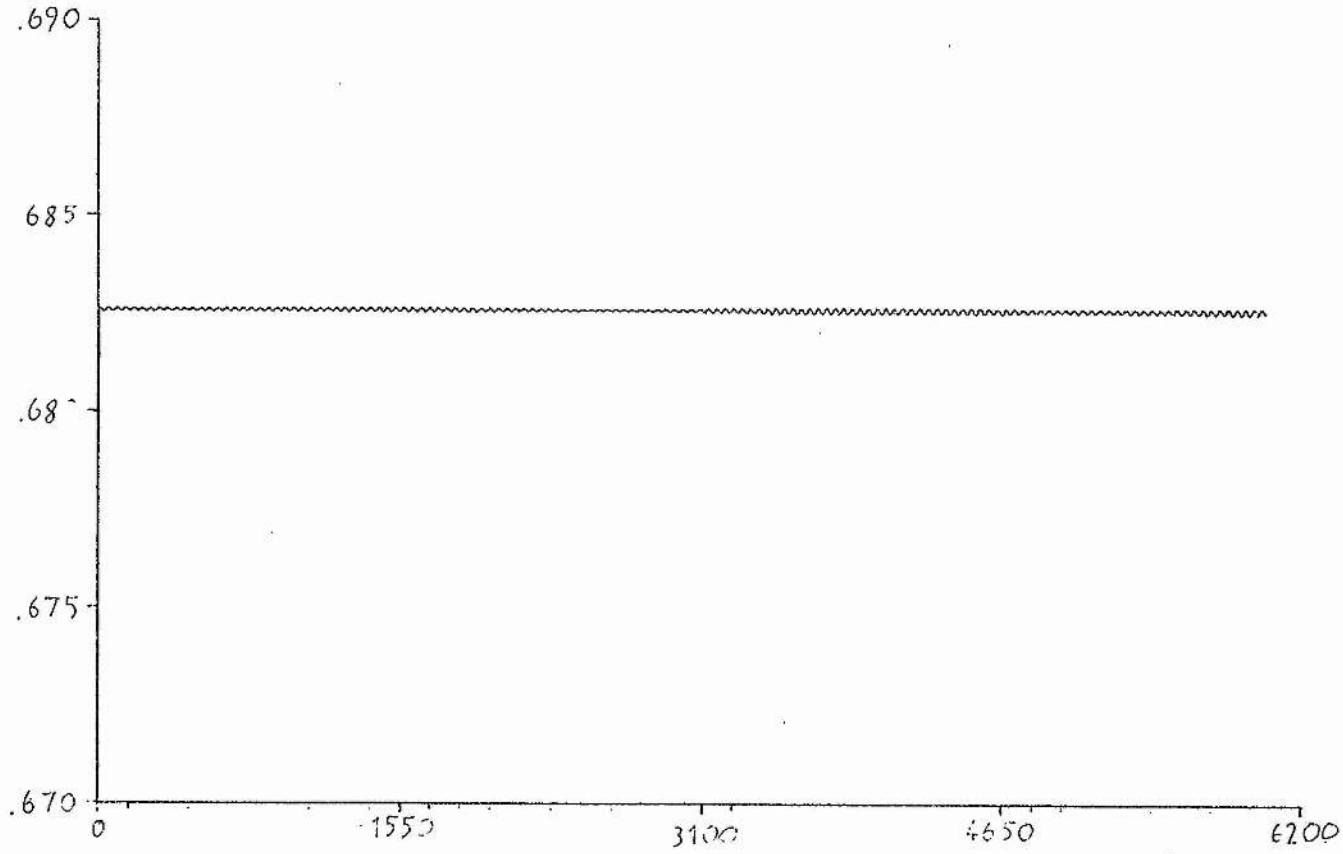
The orbit of (414) Liriope was integrated over a period of 1507 time-units, during which time the semi-major axis remains near 0.675 units (3.51 A.U.), fluctuating only slightly, with a period of around 250 units. The amplitude of this fluctuation varies between 0.13% and only 0.02% of the average value. This variation may be periodic, but there are insufficient data to confirm this or to ascertain its period.

The eccentricity exhibits oscillations of the same period as those of the semi-major axis, but opposite in phase, and varying in amplitude in the same way, between 3% and 6% of the average value. During the interval of the integration, it decreases from 0.076 to 0.024.

The curve of ψ_2 against ψ_1 suggests a circle with about five cusps, only one of which is seen; there is insufficient of this curve to warrant an attempt to fit a circle to it. The perihelion advances from 52 degrees longitude to 149 degrees (a rate of 0.034 degrees/year), passing through 90 degrees at t = 700 units.

The inclination has small oscillations similar to those in the eccentricity and the semi-major axis, but of slightly longer period - about 260 time units - and fairly constant amplitude of about 0.06 degrees (about 0.7% of the average value). Over this interval it decreases from 9.5 degrees to 8.5 degrees. The curve of χ_2 against χ_1 is a good fit to a circle of radius 0.15 centred on (0, 0.02), correponding to an average inclination of 8.6 degrees with an offset of about 1 degree. The ascending node retrogresses from 99 degrees to 8 degrees (a rate of 0.032 degrees/year), passing through 90 degrees at t = 150; the node and the perihelion thus move at approximately the same speed in opposite directions. (Although they traverse almost the same angle, it is much easier to fit a circle to the χ-curve than to the ψ-curve, because the former has no cusps.)

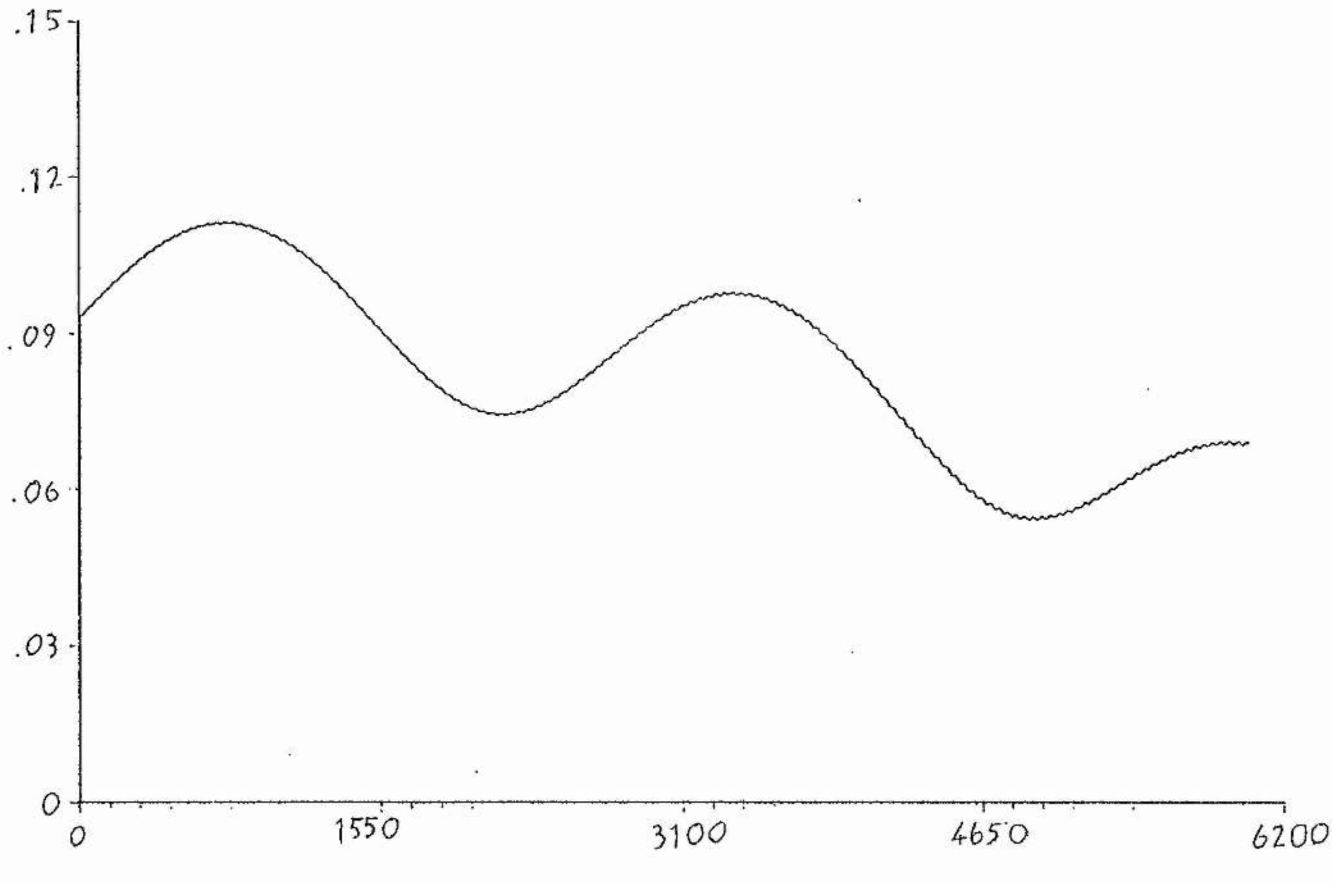
The critical angle σ is found to decrease monotonically, at the fairly slow rate of 0.2 degrees/year, throughout the integration; no approach nearer than 0.25 units (1.3 A.U.) occurs.



(909) Ulla

$$\frac{p+q}{q} = \frac{9}{5}$$

a against t

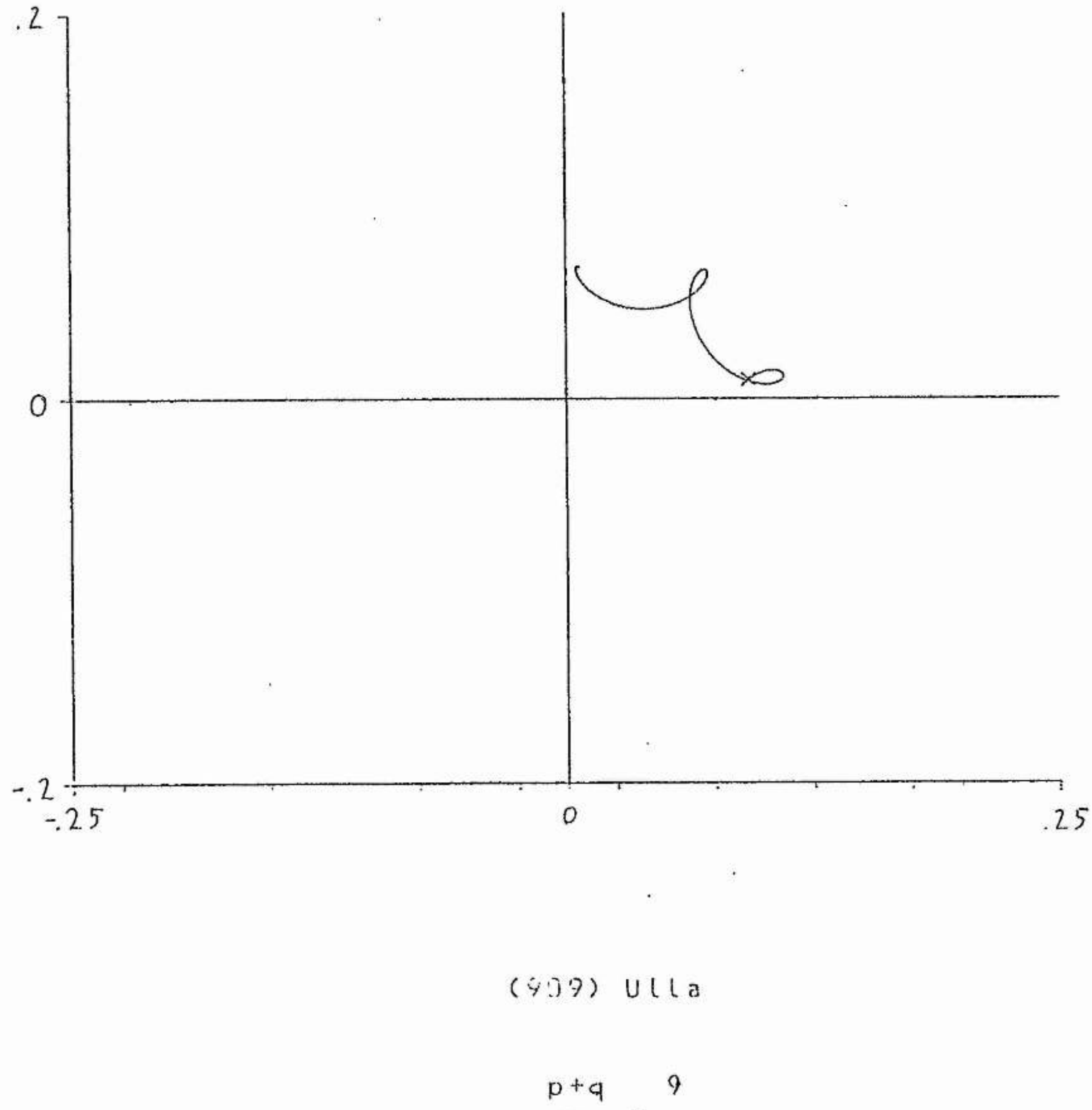


(909) Ulla

$$\frac{p+q}{q} = \frac{9}{5}$$

e against t

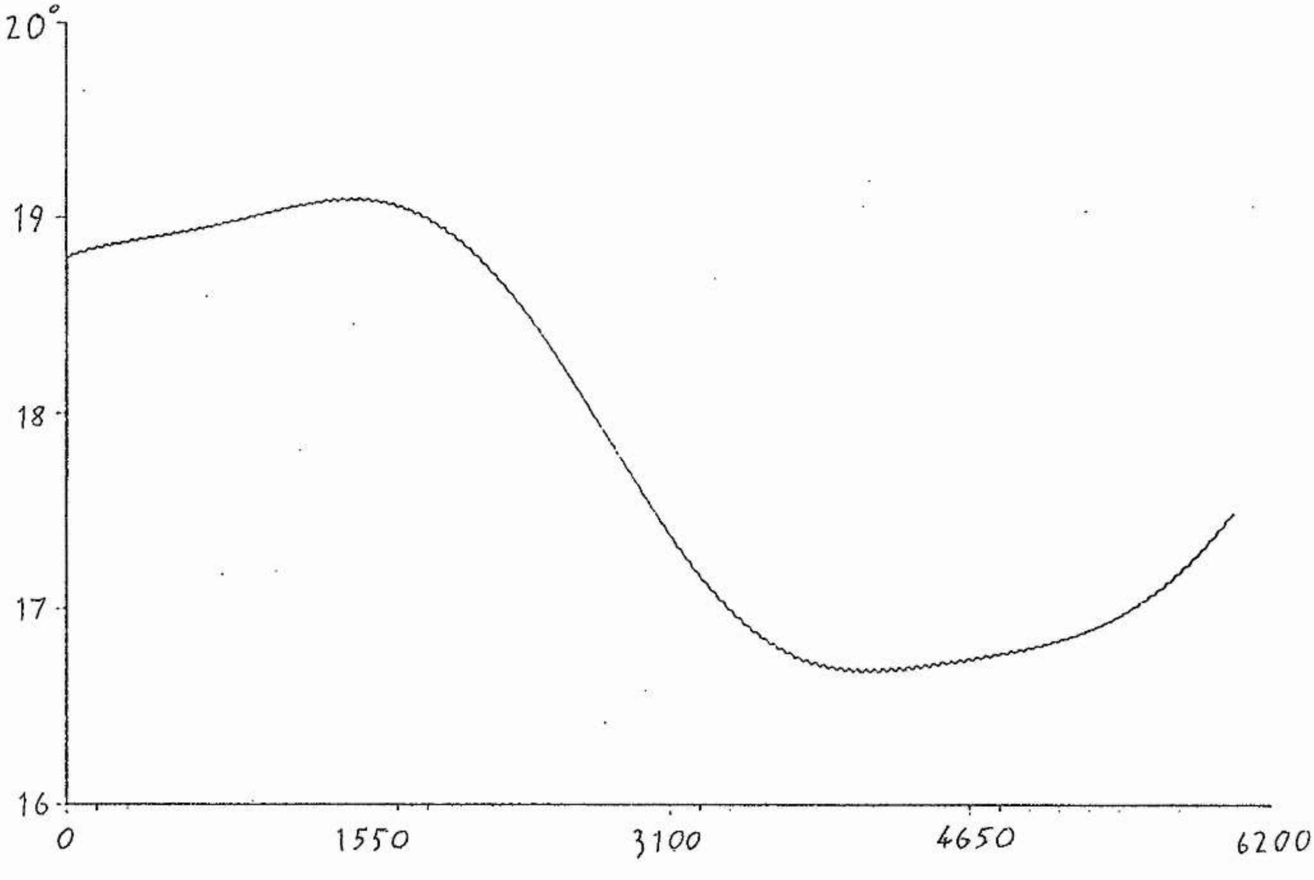
curve begins at X



(909) Ulla

$$\frac{p+q}{q} = \frac{9}{5}$$

ψ_2 against ψ_1

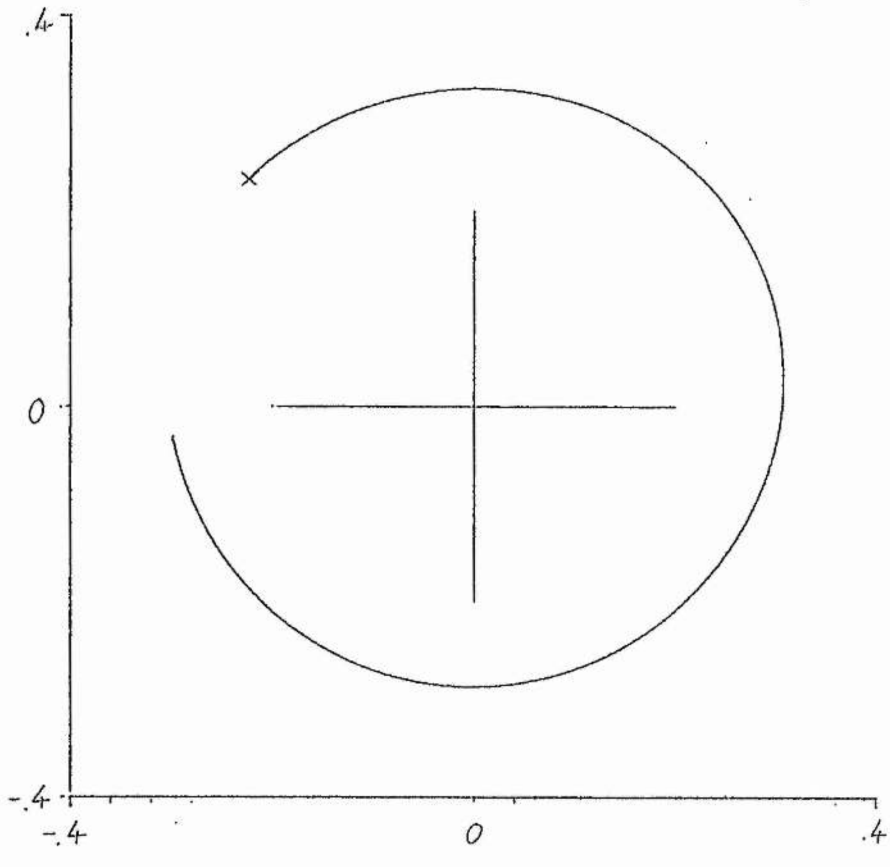


(909) Ulla

$$\frac{p+q}{q} = \frac{9}{5}$$

i against t

curve begins at X



(909) Ulla

$$\frac{p+q}{q} = \frac{9}{5}$$

χ_2 against χ_1

The first integration of the orbit of (909) Ulla was carried out with values of p and q corresponding to a ratio of mean motions of 9/5. This was continued for several nights' computing; the graphs shown here cover an interval of 6015 time-units.

The semi-major axis remains close to 0.683 units, or about 3.55 A.U. It carries very small fluctuations, which have a period of about 43 time-units, and which vary in amplitude between 0.004% and 0.02% of the average value. This amplitude variation appears to be periodic; the period is ill-defined, but may be around 3000 time-units.

The eccentricity also has small oscillations, which have the same period and the same phase as those in the semi-major axis. They appear, however, to be of constant amplitude - around 0.3% of the average value. They are superimposed on a larger fluctuation, with period of about 2400 time-units, and amplitude of 0.015 (about 18%); the eccentricity varies between 0.11 and 0.06 during the period of the integration.

The graph of ψ_2 against ψ_1 shows that the perihelion advances, with retrograde epicycles corresponding to the maxima in the eccentricity. Three of these epicycles can be seen, with a separation of about 40 degrees in the longitude of the perihelion. Over the period of the integration, the perihelion advances from 6 to 84 degrees, moving at a rate of 0.01 degrees/year.

The inclination shows small fluctuations, of the same period as those in the semi-major axis and the eccentricity, but opposite in phase; they have a fairly constant amplitude of about 0.05% of the average value. The inclination also varies slowly, between 19 and 16.5 degrees. The graph of X_2 against X_1 is a close approximation to a circle of radius 0.31, centred on (0, 0.02) - corresponding to a constant inclination of 17.9 degrees offset by 1.2 degrees. During the integration, the ascending node retrogresses from 133 to 186 degrees, at an average rate of 0.03 degrees/year. The node thus moves nearly three times as fast as the net motion of the perihelion. It passes through the values of 90 degrees, zero and 270 degrees longitude at 890, 2760 and 4430 time-units.

The critical angle σ is found to increase monotonically throughout the integration, at about one degree a year. The minor planet never comes closer to Jupiter than 0.2445 units - about 1.27 A.U.

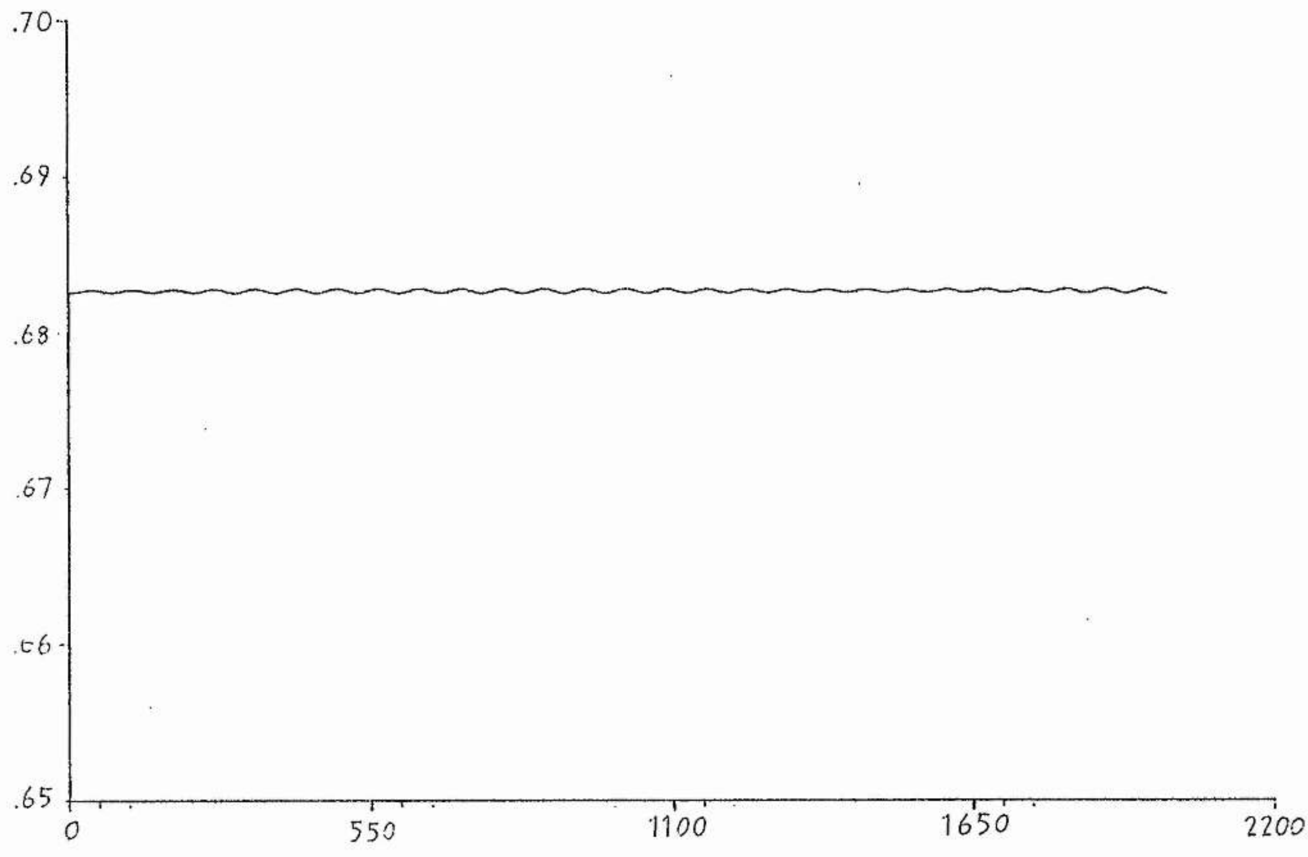
At an earlier stage, this orbit integration was carried out for an even longer period, covering an interval of almost 10,000 time-units, or about 189 centuries. Unfortunately, the results of this integration were not saved in reproducible form. In general, however, the orbital parameters were found to continue to behave in much the same way as shown here.

The semi-major axis continued to show variations in the amplitude of its oscillations; these had a period of around 3000 time-units. The maximum amplitudes in each cycle, however, grew progressively larger; by the end of the integration they had reached 0.04% of the average value. It could not be determined whether this was part of a larger cycle of variations, or whether the accumulation of numerical errors over so many integration-steps had finally become visible as a secular change in the orbit.

The graph of ψ_2 against ψ_1 continued around the circle with a fourth epicycle, the perihelion reaching a longitude of 235 degrees by the end of the integration. This circle was centred at about (0.03, 0), with a radius of 0.06. The epicycles grew progressively smaller, both in length and in width, showing that the variations in both the eccentricity and the longitude of perihelion were reducing in magnitude. Again, it was not clear whether this was a cyclic phenomenon - they might increase again to their original size as the circle was completed - or a secular variation due to numerical errors accumulating. However, if the latter explanation is the true one, it is interesting that the accumulated errors act in the opposite direction on the two curves, increasing the instability of one and decreasing the instability of the other.

The inclination continued to vary slowly between the same limits, and the graph of

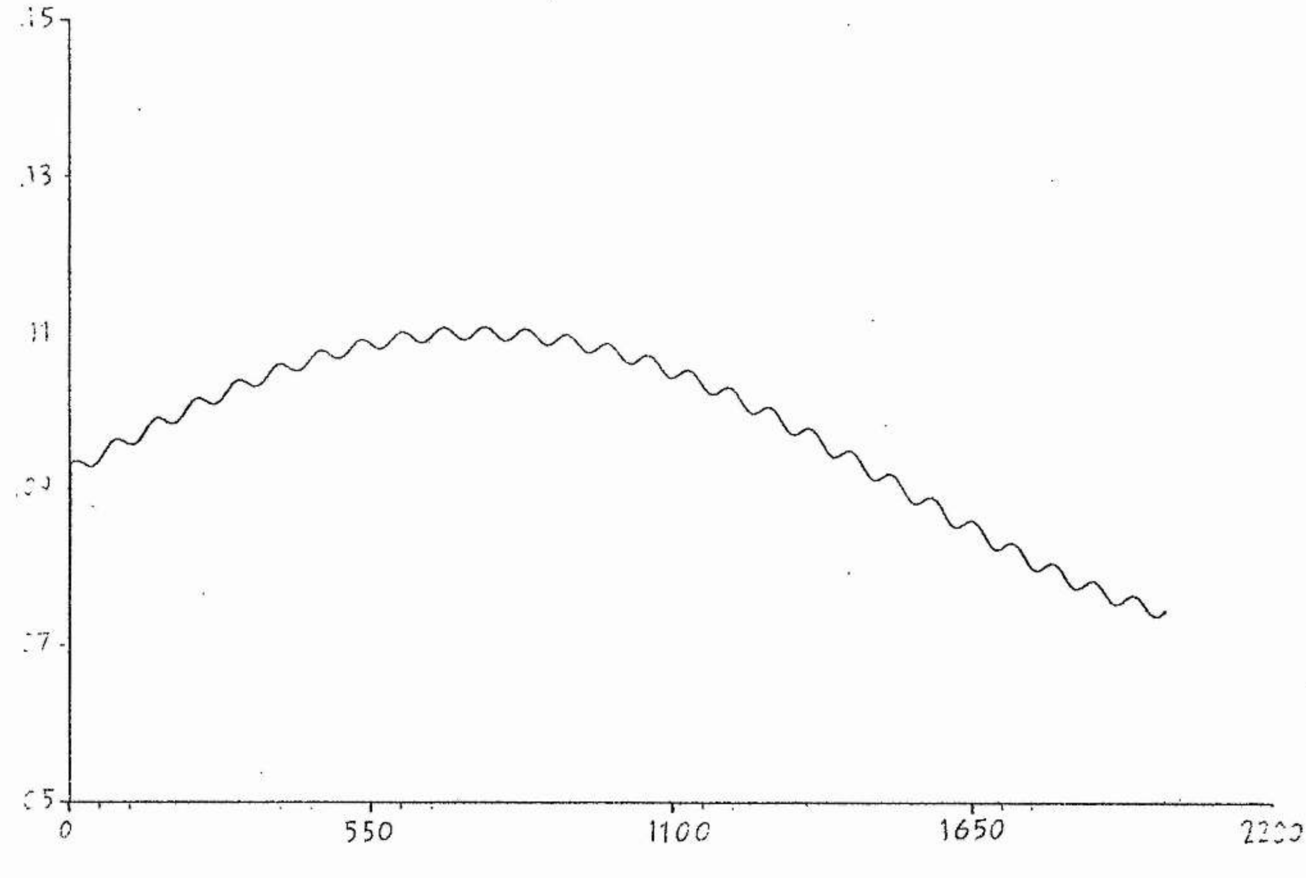
X_2 against X_1 continued to trace out the same circle, to a total of almost one and a half revolutions.



(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

a against t

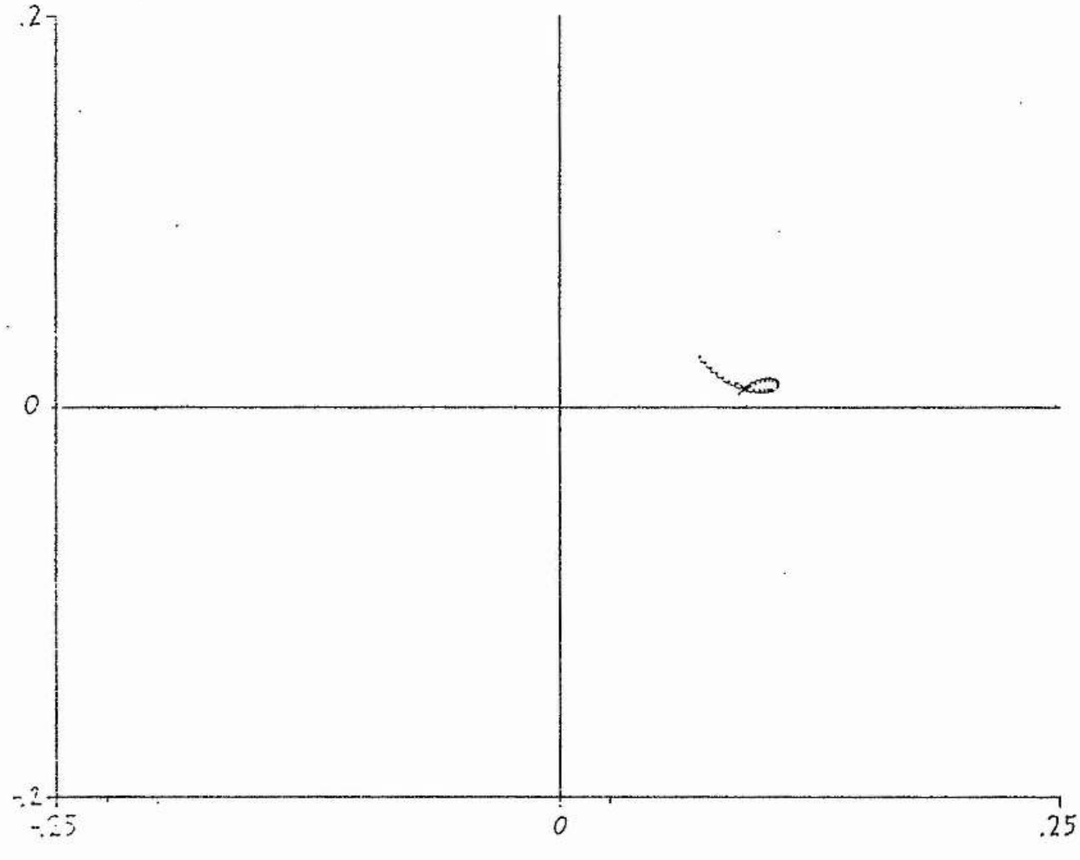


(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

e against t

curve begins at X

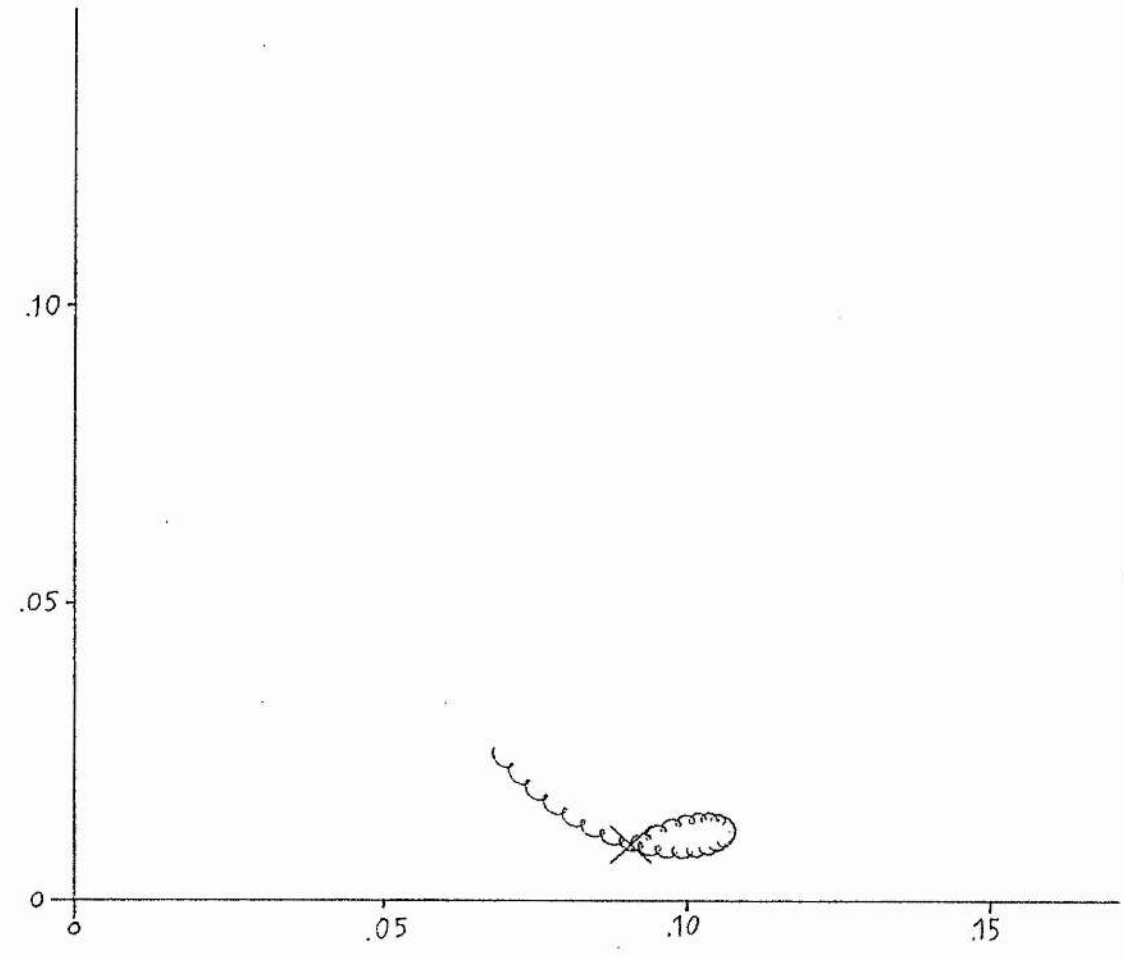


(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

ψ_2 against ψ_1

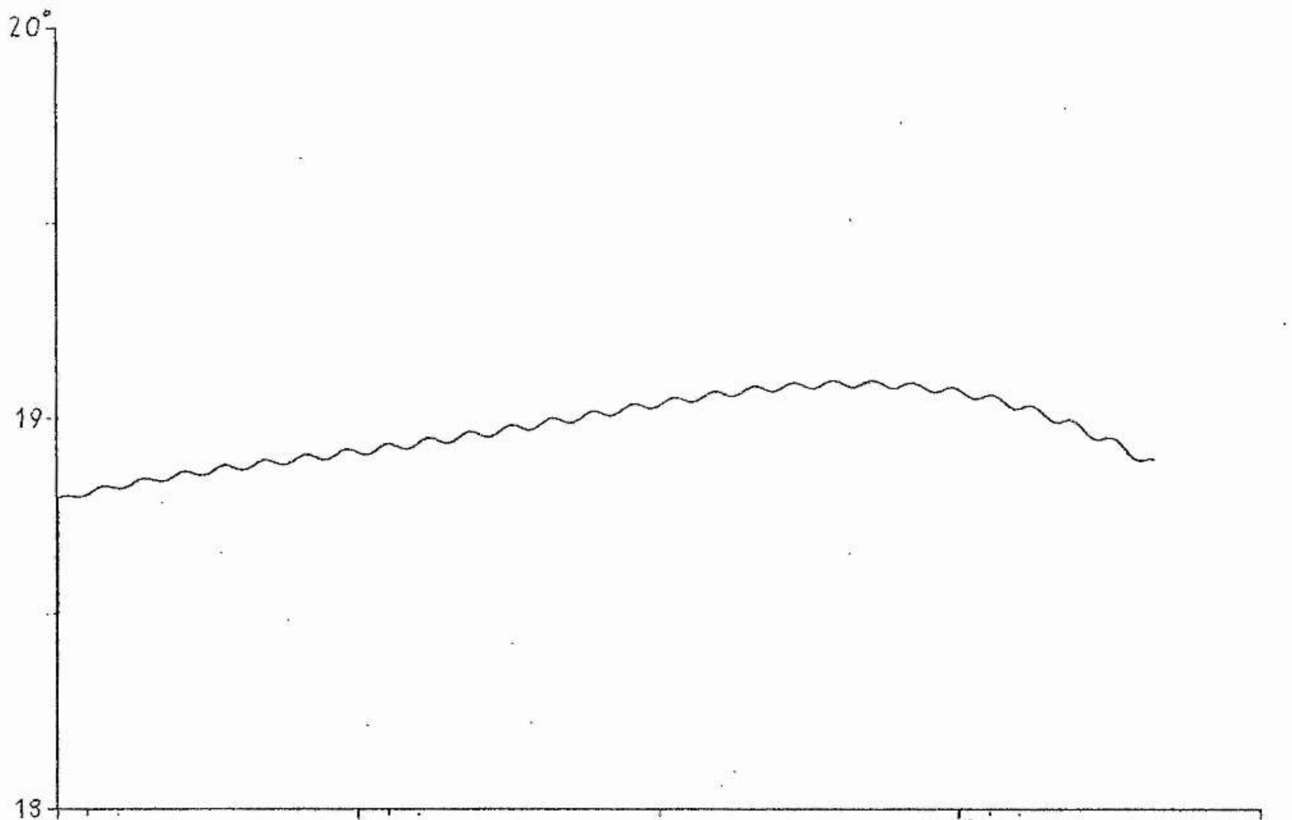
curve begins at X



(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

ψ_2 against ψ_1

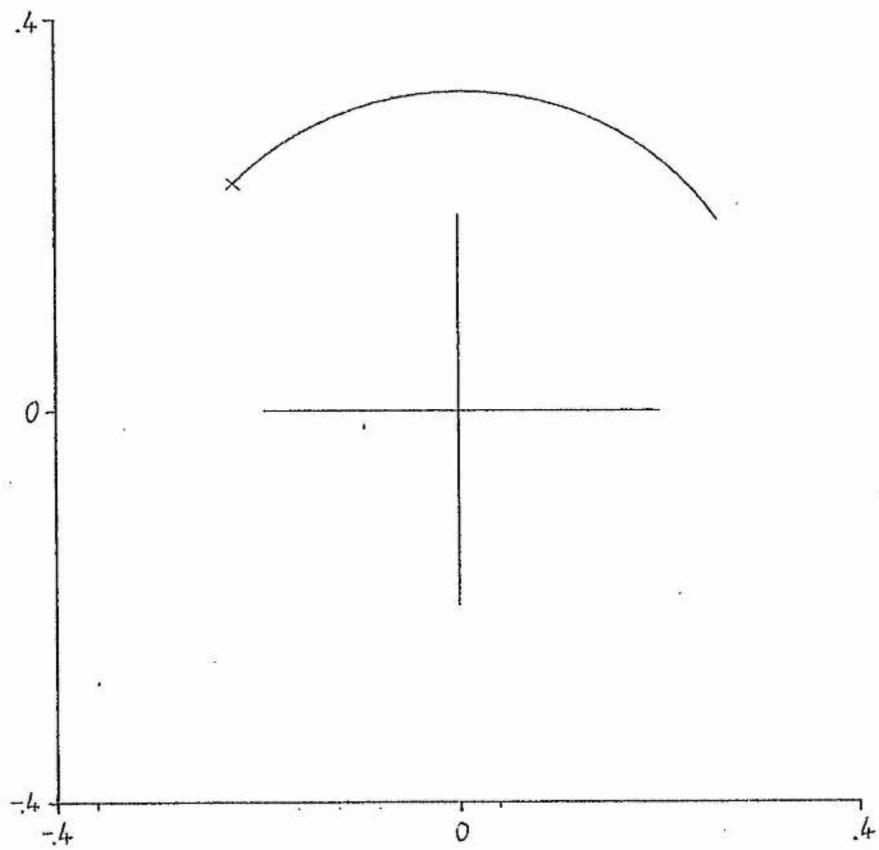


(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

i against t

curve begins at X



(909) Ulla

$$\frac{p+q}{q} = \frac{7}{4}$$

χ_2 against χ_1

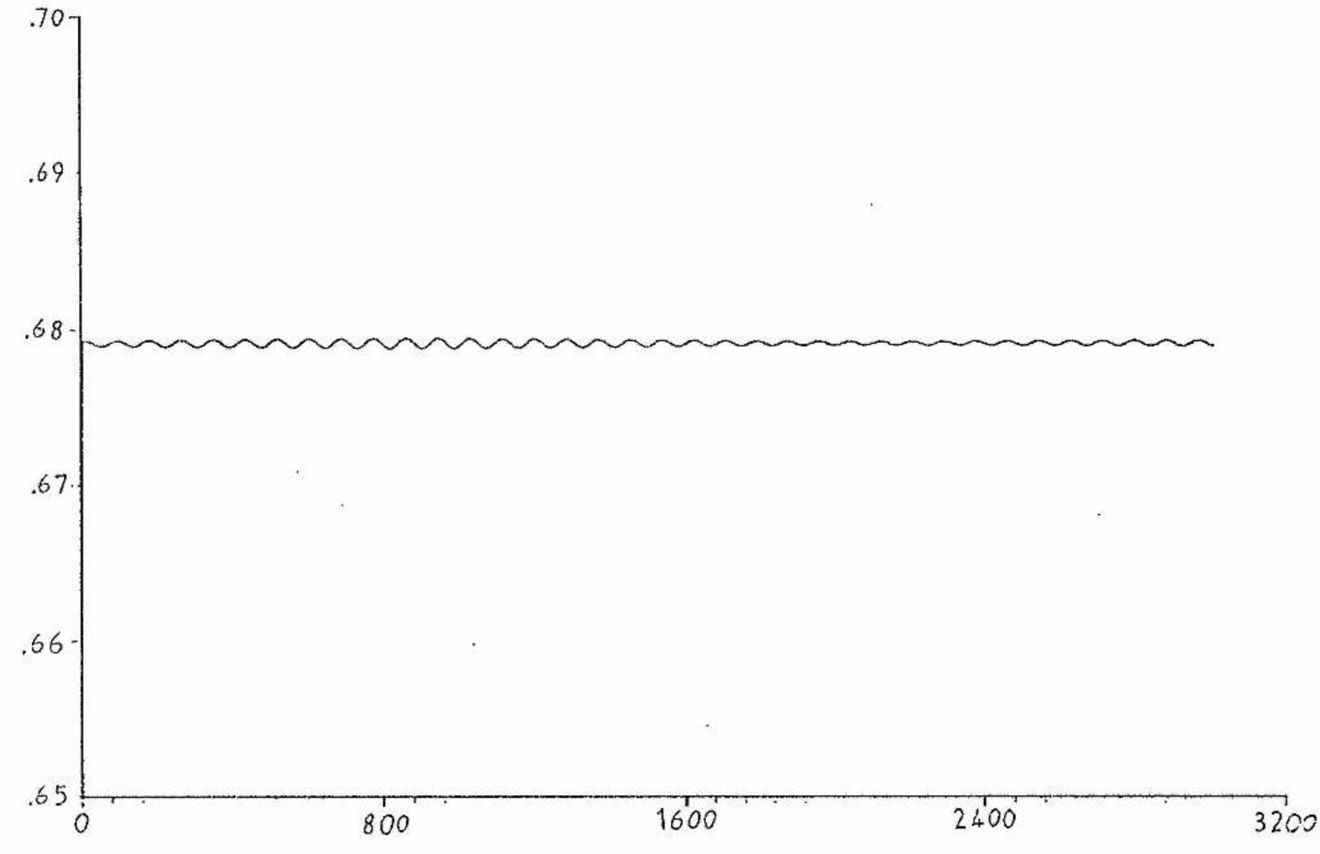
This integration of the orbit of (909) Ulla was carried out with the ratio of mean motions taken to be 7/4; this reduction in the values of p and q enabled the integration to be carried slightly further than normal in one night's computing - to 2007 time-units, or 3800 years.

Over this period, the semi-major axis remains nearly constant at 0.68 units (3.54 A.U.), with fluctuations of about 0.03% at a period of 75 time-units. The eccentricity has fluctuations of the same period and opposite phase, with a somewhat larger relative amplitude - about 2% of the average value. It also undergoes a slow variation between 0.07 and 0.11 during the period of the integration.

The graph of ψ_2 against ψ_1 is repeated on the following page, at a larger scale, so that the details can be distinguished. The small oscillations in eccentricity combine with small advances and retreats of the perihelion, to give retrograde epicycles. The larger variation in eccentricity similarly produces larger epicycles, again retrograde. However the overall motion of the perihelion is a slow advance; it moves from 6 degrees longitude to 21 degrees over the period of the integration, a net motion of only 0.004 degrees/year.

The inclination ranges between 18.8 and 19.1 degrees; it also has a small oscillation, of period 74 time-units, which is very close to, but not identical with, the period of the oscillations in the semi-major axis and the eccentricity. The curve of χ_2 against χ_1 is a good fit to a circle with centre at the centre of the plot, or nearly so, and radius of about 0.32 (corresponding to an inclination of 18.8 degrees). The ascending node decreases from 134 to 37 degrees over the interval of the integration - a rate of about 0.03 degrees/year, which is considerably faster than the net movement of the perihelion. The node passes through 90 degrees longitude at a time of 895 time-units.

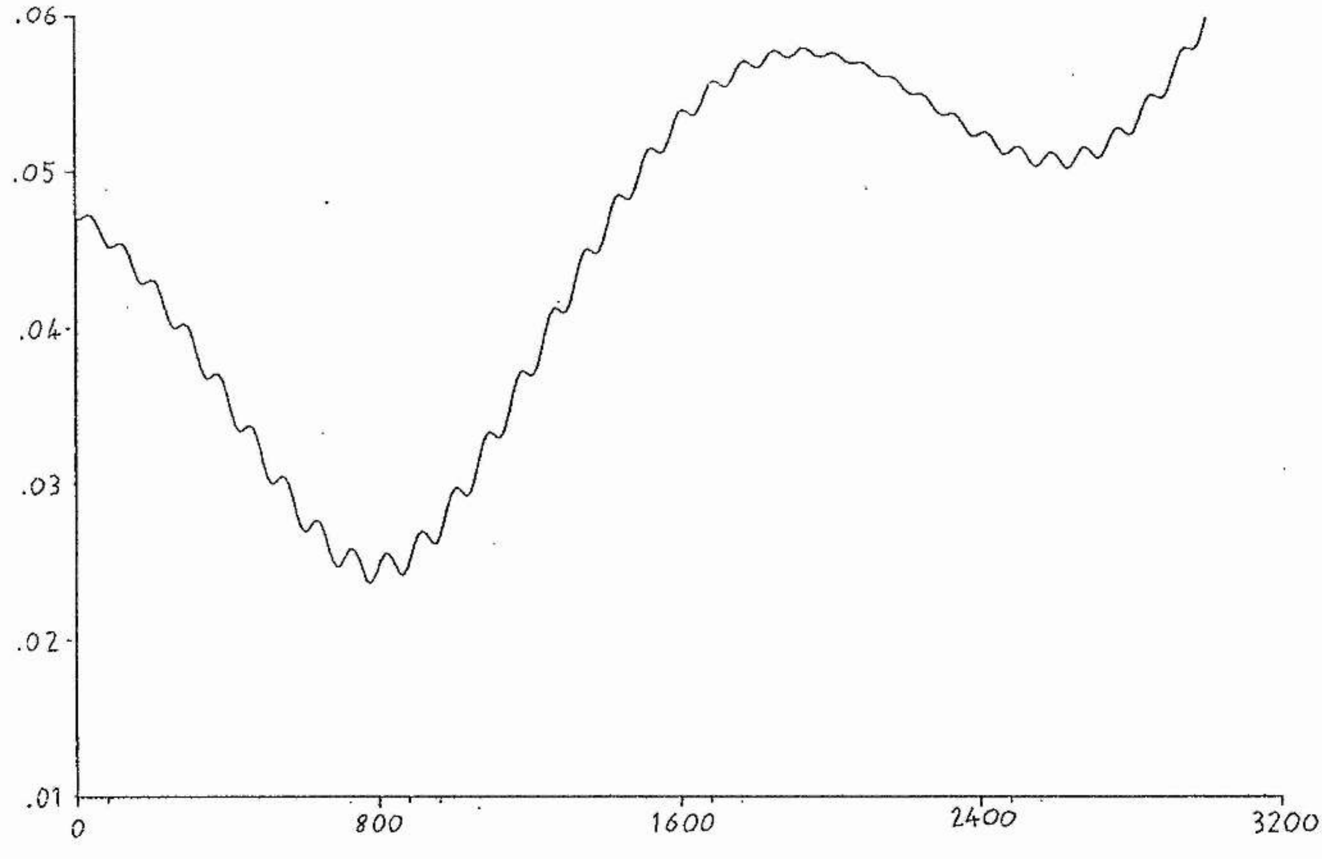
The critical angle σ decreases monotonically throughout the period of the integration, at a rate of nearly a degree a year; no approach to Jupiter closer than 0.25 units (about 1.3 A.U.) occurs.



(1574) Meyer

$$\frac{p+q}{q} = \frac{9}{5}$$

a against t

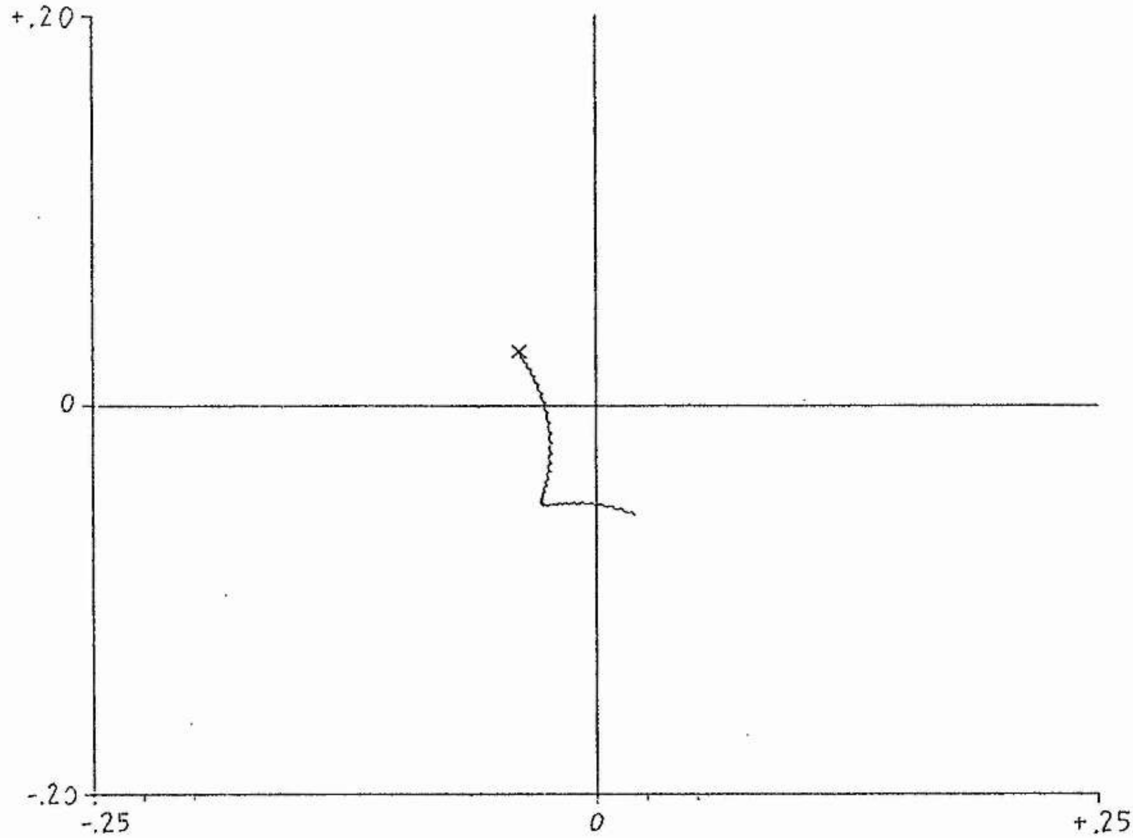


(1574) Meyer

$$\frac{p+q}{q} = \frac{9}{5}$$

e against t

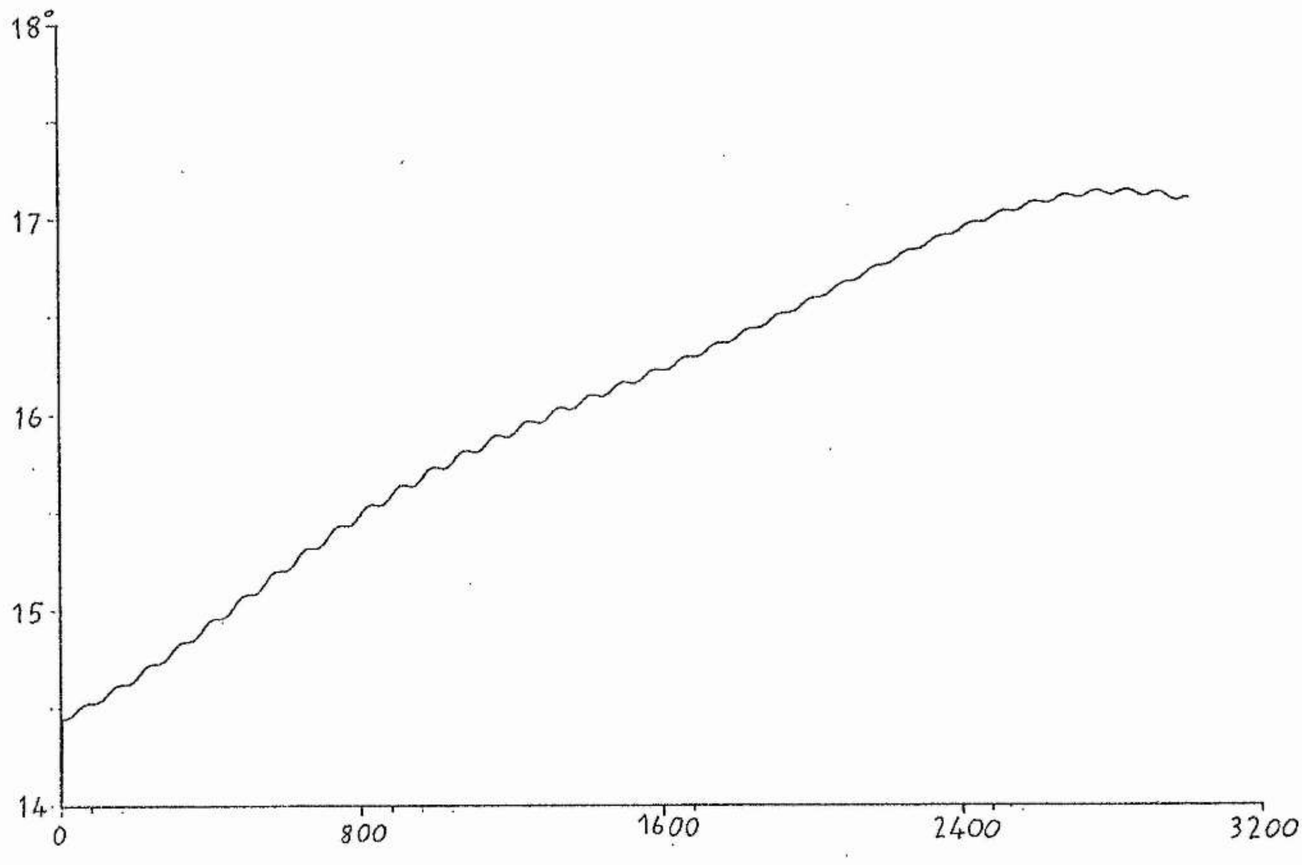
curve begins at X



(1574) Meyer

$$\frac{p+q}{q} = \frac{9}{5}$$

ψ_2 against ψ_1

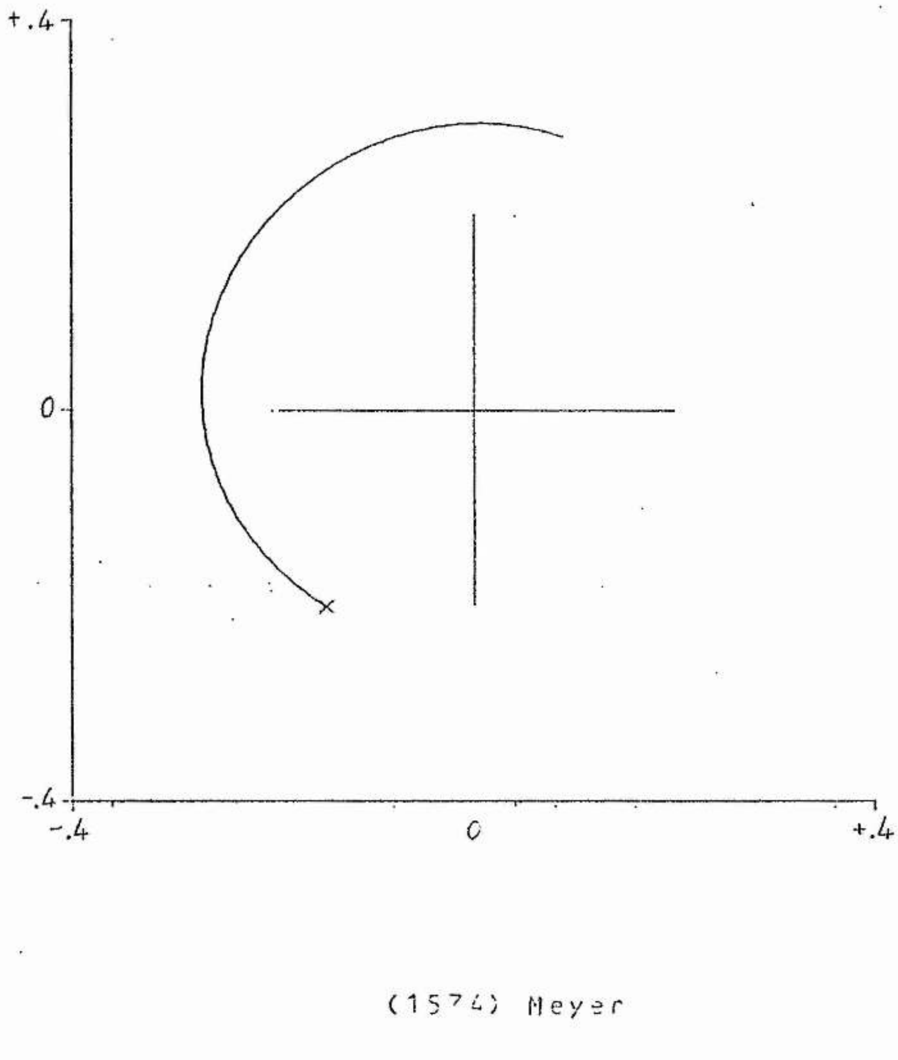


(1574) Meyer

$$\frac{p+q}{q} = \frac{9}{5}$$

i against t

curve begins at X



(1574) Meyer

$$\frac{p+q}{q} = \frac{9}{5}$$

X_2 against X_1

The orbit of (1574) Meyer was integrated over a period of 3010 time-units, or about 5700 years. This was carried out in two successive nights' computing, and all the results were kept and plotted together, in the graphs reproduced here.

The semi-major axis remains close to 0.68 units, or about 3.54 A.U. It undergoes small oscillations, with a period of 85 time-units; these vary in amplitude, from 0.04% to 0.09% of the average value, over a period of about 3000 time-units.

The eccentricity shows similar oscillations, of the same period but opposite in phase. Although these are superimposed on a larger oscillation, it is nevertheless possible to see that the amplitudes of the small oscillation vary in the same way as those in the semi-major axis; they are greatest (about 4% of the average value) when the eccentricity is at a minimum, and least (1.5%) after it has passed its maximum, showing that the two cycles are not directly related. Overall, the eccentricity varies between 0.02 and 0.06 over the period considered, and is still increasing at the end of the period.

The first half of the graph of ψ_2 against ψ_1 was unexpected, as it looked like a section of a circle centred well away from the centre of the plot; this was the reason for extending the integration for a second night. The finished graph indicates that the curve is probably an approximation to a circle centred at about

(0.036, 0), with a radius of 0.068; however, there is not sufficient of the curve for these figures to be accurate. There is one definite cusp, corresponding to a maximum in the eccentricity, but it is not possible to see what the spacing between cusps will be. Over the interval covered by the integration, the perihelion advances from 144 to 289 degrees - about 0.025 degrees/year. It passes through 180 and 270 degrees at 722 and 2678 time-units.

The inclination shows a small oscillation which appears to be of exactly the same period as those in the semi-major axis and the eccentricity; it is in the same phase as the eccentricity. It is modulated in amplitude in the same way, varying between 0.3% and 0.4% of the average value. The inclination also undergoes a larger, slower, fluctuation, which varies it between 14.5 and 17 degrees; the period of this fluctuation is undefined from this graph.

The curve of χ_2 against χ_1 is a good fit to a circle with centre at (0, 0.024), and radius 0.272, corresponding to an inclination of 15.8 degrees, offset by 1.4 degrees. The ascending node retrogresses from 234 to 72 degrees longitude, travelling a little more slowly than the perihelion, at 0.028 degrees/year. It passes through 180 and 90 degrees at 946 and 2651 time-units.

Over the period of the integration, the critical angle σ increases monotonically, at about half a degree a year. The minor planet never comes closer to Jupiter than 0.25 units - 1.3 A.U.

It is possible to compare directly the results for five of these integrations - those for (87) Sylvia, (107) Camilla, (414) Liriope, (1574) Meyer, and (909) Ulla (with a ratio of 9/5). This list is in order of increasing semi-major axis, and the first point to be noticed is that the rate of change of the critical angle σ varies steadily from -1 to +1 degrees/year when the planets are arranged in this order. It changes sign between (414) Liriope and (1574) Meyer, which corresponds to the exact position of the commensurability.

The slowest movement of σ occurs for (414) Liriope: 0.2 degrees/year. The mean motion of this planet is the nearest to 1.80 times that of Jupiter; the difference is 0.5%. By comparison, the angle σ oscillates in the case of (153) Hilda, where the ratio of mean motions differs from 1.50 by 0.2%. For (909) Ulla, the ratio of mean motions is slightly less different from 7/4 (1.3%) than from 9/5 (1.5%); the change in σ is about 1 degree/year in each case.

The rates of movement of the ascending nodes of the five orbits are roughly similar; they retrogress at about 0.03 degrees/year, completing a revolution in about 6000 years. The rates of movement of the perihelion vary much more widely, but this may be related to the much more pronounced cusps on the ψ-curves than on the χ-curves. The net movement is smallest for (909) Ulla, in which the cusps extend to definite epicycles, with the perihelion actually reversing direction. The pointed cusp on the ψ-curve for (1574) Meyer also indicates a slowing-down, almost to standstill, of the perihelion. The curves for (107) Camilla and (414) Liriope, on the other hand, suggest a fairly steady movement of perihelion, and these two orbits in fact have a rate of change of perihelion slightly greater than that of the ascending node. The exception to this pattern is (87) Sylvia, where the ψ-curve is simple, but the rate of change of perihelion is little over half that of the node.

The χ-curves are all much simpler than the ψ-curves, generally showing only the small-scale oscillations, and being otherwise close approximations to circles. The centres of these circles all appear to lie on the χ_2-axis, and are all displaced by similar amounts. (The centre of the circle for the 7/4 integration of (909) Ulla is estimated as being at the intersection of the axes, i.e. as having no displacement; but in fact the quarter-circle available corresponds very closely to the circle in the 9/5 integration, which has a displaced centre.) The radii of the circles correlate closely, as would be expected, with the inclinations of the starting orbits.

The ψ-curves are much less straightforward. They all show more or less large-scale fluctuations in radius, making the fitting of a circle largely a matter of guesswork. The centres of these circles all appear to lie on the ψ_1-axis, but the displacement varies considerably. However, four of the graphs given are not inconsistent with a common centre at (0.04, 0); the exception again is (87) Sylvia, where the centre is definitely much nearer the intersection of the axes. The average radii of the circles do not correlate with the eccentricities of the starting-orbits; this is undoubtedly because the radii fluctuate, and the starting-values may lie anywhere within the range of these fluctuations.

The periods of the small-scale oscillations vary widely, from 80 to 470 years, and do not appear to correlate with any of the parameters of the orbits. In all the cases considered, the oscillations in the semi-major axis and the eccentricity have the same period, as closely as can be determined; but, in every case except the 9/5 integration of (909) Ulla, they are opposite in phase. In the latter case, and also in (1574) Meyer, the oscillations in the inclination also appear to have exactly the same period, in opposite phase to the semi-major axis. For the two planets next nearer the sun the periods of the oscillations in the inclination are slightly greater than of those in the first two parameters - 1.04 times and 1.06 times respectively. For the planet nearest the sun, (87) Sylvia, the oscillation in the inclination is on such a small scale the the effects cannot be separated from those of the long-period variation; all that can be said is that is period is within 10% of that of the first two parameters.

The variations in the orbit of (334) Chicago, taken relative to the 3/2 commensurability, are by far the greatest of those considered here. This planet has a ratio of mean motion to that of Jupiter of 1.549, and consequently is in fact twice as far removed from the chosen commensurability as any of the others. The next commensurability nearer to the actual position of (334) Chicago is 14/9, which was not tried here. Schubart (1968) suggests that the ψ-curve of this planet, which shows a slow advance of perihelion combined with retrograde epicycles, is typical of a "transition" case between commensurable and non-commensurable motion. This suggestion is supported by the corresponding curves for (909) Ulla which show smaller epicycles, for both the 9/5 and 7/4 commensurabilities; this planet is nearer to both these commensurabilities than (334) Chicago is to 3/2, but further than any of the remaining planets.

The integration of the orbit of (153) Hilda shows much larger-amplitude and shorter-period oscillations in the semi-major axis and the eccentricity than in the other orbits considered. The movement of the perihelion, besides being in the opposite direction (being monotonically retrograde), is also much faster, and the centre of the ψ-curve is more displaced from the intersection of the axes. The period of the oscillations in inclination falls within the range of the others considered, but the ampitude is much larger and the shape more complex than for any other planet except (334) Chicago. The ascending node however moves at much the same speed as those of the 9/5 planets. (153) Hilda has a larger semi-major axis (3.98 A.U.) and larger eccentricity (0.154) than the other planets, and a smaller inclination than most (7.8 degrees), but it is not clear which factors are responsible for the various differences.

It is interesting to compare the results of the two integrations on the orbit of (909) Ulla, using different values of p and q. On the whole, the graphs are remarkably similar, the various parameters of the orbit varying in the same way between the same limits. The details of the small-scale oscillations, however, do differ.

The periods of the oscillations in the semi-major axis are 43 time-units for the ratio 9/5, and 75 for the ratio 7/4. In the latter case, no change in the amplitude of these oscillations is noticeable - it remains about 0.03% of the average value. In the former case, the amplitude varies, over a period of several hundred time-units, between 0.004% and 0.04%.

The oscillations in the eccentricity are in phase with those in the semi-major axis, for the ratio 9/5; for the ratio 7/4, as indeed in all the other integrations, they are out of phase. They are also considerably smaller in the former case. The graphs of ψ_2 against ψ_1 show very similar large epicycles in both cases, but only the 7/4 ratio exhibits small epicycles as well; the 9/5 curve has only small cusps. This suggests that the longitude of perihelion does not actually reverse direction in the case of 9/5, but probably slows down and speeds up as the eccentricity varies; as already observed, the variations in the eccentricity are themselves smaller. The perihelion moves much more quickly overall for the ratio 9/5 -

about three times as fast. This may be partially due to the extra back-and-forth motion covered in the 7/4 case; the change in longitude over one time-unit varies, but is roughly similar in each case.

The variations in the inclination have the same period as those in the semi-major axis and the inclination, for the ratio 9/5; they are slightly different for 7/4. The curves of χ_2 against χ_1 are almost exactly the same circle in both cases, and the ascending node moves at approximately the same speed.

The critical angle σ increases steadily when the ratio is taken as 9/5; it decreases, at about the same rate, when the ratio is 7/4.

Thus it would seem that, in its small-scale details, the integration of the orbit of (909) Ulla with a ratio of 9/5 shows certain differences from most of the other orbits considered; but, in the longer term, the behaviour of the orbit is much the same, no matter which ratio is assumed for the mean motions.

At one stage in the calculations, owing to an error, several integrations were carried out with the argument of the perihelion ω in place of its longitude ϖ. This amounts to an arbitrary alteration of the longitude of the perihelion; consequently, though the results have no correspondence with any physical orbit, they do cast some light on the stability of the method against random variations in the starting parameters.

The most dramatic difference was observed in the orbit of (153) Hilda, where two of the curves appeared completely different. On closer examination, it was seen that this was due almost entirely to a change in the amplitude of the short-period oscillations in the eccentricity - they were three or four times larger than they ought to be. Their period, however, was almost unchanged. The ascending node retrogressed considerably faster with the erroneous value of the longitude, but the other results were very similar.

In the case of (107) Camilla, on the other hand, the small oscillations had a much smaller amplitude when the wrong longitude was used. The ascending node moved a little more slowly than it ought, and the perihelion moved at only about half the correct speed (but in the right direction).

For (909) Ulla, with the ratio of mean motions taken as 9/5, the only change was in the small oscillations in the inclination, which came out too large and with a slightly shorter period.

Apart from the changes described, the integrated orbits of these three planets showed considerable consistency when the longitudes of perihelion were changed. All the orbital parameters varied with time in the same sort of way, and remained within very similar limits. This suggests that the method used is probably reasonably stable against random changes in the starting parameters.

It is now possible to compare the orbits obtained from observations, in section II, with those predicted by numerical integration, in this section, for four of the planets investigated. (The planet (153) Hilda was not observed; no orbit was obtained for (1574) Meyer; and the orbits obtained for (334) Chicago were not regarded as reliable.)

We may tabulate the results for each planet, together with the original parameters from the "Ephemerides of Minor Planets", as follows.

(87) Sylvia

	original	predicted	observed
a	3.4812	3.4812	3.4816
e	0.099	0.099	0.094
ω	$265^\circ 33'$	$266^\circ\ 1'$	$272^\circ\ 2'$
i	10 51	10 51	10 54
Ω	74 4	73 45	73 22

Here the semi-major axis, the eccentricity and the inclination have all varied more than the theory predicts; in fact the numerical integration indicates that none of these parameters should ever reach these values. The perihelion and the ascending node have both moved in the expected direction, but more quickly; the node has moved twice as fast, and the perihelion nearly fourteen times as fast.

(107) Camilla

	original	predicted	observed
a	3.4895	3.4392	3.4888
e	0.070	0.069	0.073
ω	303°34′	305°56′	291°59′
i	9 55	9 58	9 57
Ω	174 40	173 45	174 3

Here again the semi-major axis and eccentricity have changed more than predicted, although the eccentricity is not out of the range predicted by the numerical integration over several thousand years. The inclination has actually varied less than predicted. The perihelion has apparently retrogressed instead of advancing, and that nearly five times as fast as predicted. The ascending node, on the other hand, has moved in the predicted direction at only two-thirds the predicted speed.

(414) Liriope

	original	predicted	observed
a	3.5072	3.5071	3.4988
e	0.077	0.077	0.087
ω	313°45′	314°24′	317°13′
i	9 33	9 33	9 32
Ω	111 53	111 33	111 20

Once again the orbital parameters have varied more than predicted - especially the eccentricity, which has increased to a value the integration would not predict until after an interval of nearly 9000 years. The perihelion and the ascending node have moved in the predicted direction, but several times faster than predicted.

(909) Ulla

We have here two sets of predictions (the integrations taking the ratios of the mean motions as 9/5 and as 7/4) and three sets of results (the new orbit published in "Ephemerides of Minor Planets", and the two orbits derived from observations in this investigation).

We give each set of results separately.

	original	predicted(9/5)	predicted(7/4)	E.M.P.
a	3.5513	3.5514	3.5513	3.5453
e	0.093	0.093	0.093	0.091
ω	232° 6′	232°18′	232°10′	229° 2′
i	18 48	18 48	18 48	18 45
Ω	147 12	147 3	147 3	146 58

	original	predicted(9/5)	predicted(7/4)	observed(1)
a	3.5513	3.5514	3.5513	3.5463
e	0.093	0.093	0.093	0.091
ω	232° 6′	232°25′	232°12′	227°22′
i	18 48	18 48	18 48	18 48
Ω	147 12	146 58	146 58	146 52

	original	predicted(9/5)	predicted(7/4)	observed(2)
a	3.5513	3.5514	3.5513	3.5416
e	0.093	0.093	0.093	0.092
ω	232° 6′	232°27′	232°13′	228°31′
i	18 48	18 48	18 48	18 45
Ω	147 12	146 56	146 57	146 56

The first point to be noticed here is that the differences between the two sets of predictions are much less than between the predicted and the observed values. The eccentricity and inclination do not vary outside the total range predicted by the numerical integration over a long period of time, but the semi-major axis has apparently become appreciably smaller than predicted.

Both the perihelion and the ascending node have changed direction over the period of observation. The perihelion, having retrogressed nearly five degrees in less than nine years, has started to advance again. Such behaviour is predicted to occur at various future times, when the ψ-curve is exhibiting epicycles, but the movement is larger and faster than predicted. The ascending node, having also retrogressed (a little faster than predicted) has also apparently started to advance again; this behaviour, if real, is in complete contradiction to the numerical predictions, where the node retrogresses monotonically.

It should be borne in mind that most of the changes in the orbits discussed here are fairly small, and some may be of comparable magnitude to the errors in the parameters of the orbits calculated from observations. However, the general conclusion would appear to be that the real orbits vary more, and more quickly, than the theoretical predictions would suggest.

IV - Conclusions

The general lack of close agreement between theory and observations highlights the limitations of this theoretical approach. We are treating the problem as one of only three bodies, of which the two massive ones (the sun and Jupiter) are unaffected by the third (the minor planet). This is true as far as it goes. However Jupiter does not orbit round the sun in undisturbed elliptical motion, as the theory assumes, and the minor planet is not influenced only by the sun and Jupiter. All the other planets in the solar system contribute to the real motion; in particular Saturn has a considerable effect on the orbit of Jupiter, and undoubtedly affects the minor planet as well.

The effect of Saturn on the motion is probably some orders of magnitude smaller than the primary contribution of the sun and Jupiter, and so the results of these integrations may be taken as a first approximation to the behaviour of the real orbit. In particular, all the orbits considered here appear to be stable, and it is unlikely that the effect of Saturn is large enough to alter this. It may be concluded, therefore, that all the planets considered will remain in orbits similar to their present ones, for several thousands of years. But the details of the short-period behaviour of the orbital parameters cannot safely be predicted from this theoretical analysis.

That said, the numerical integrations do present an interesting picture of the behaviour of various orbits in a "model" three-body situation.

All the orbits show an exact correspondence between the short-period oscillations in the semi-major axis and the eccentricity; they have the same period and usually opposite phase. (This means that the perihelion distance of the planet, for instance, fluctuates, not with the sum of these two oscillations, but with their geometrical mean.) One of the integrations of (909) Ulla, however, reverses this effect, the two oscillations being of the same period and the same phase, so the opposing of the phases is not an inevitable result.

The actual periods of these oscillations vary considerably from one planet to another; in each, they are closely matched by the period of the oscillations in the inclination. The behaviour of the inclinations and the nodes is, however, much simpler than that of the eccentricities and the perihelia. Each ascending node retrogresses steadily; each inclination undergoes a quasi-sinusoidal variation (which makes the graph of $\sin(i)\sin(\Omega)$ against $\sin(i)\cos(\Omega)$ a circle offset from the centre), but no other large scale fluctuations. By contrast, the eccentricities all show fluctuations superimposed on the quasi-sinusoidal curve, which, linked with variations in the movement of the perihelia, produce cusps and even epicycles on the offset circle of e $\sin(\varpi)$ against e $\cos(\varpi)$.

Finally it is possible to assess the effects of including or neglecting the inclination of the orbit in the calculations. In the case of (153) Hilda, which has an inclination of almost 8 degrees, the behaviour of the other orbital parameters is much the same whether the inclination is included or neglected.

For (909) Ulla, however, with an inclination of about 18 degrees, the effect of the inclination on the other parameters is limited but important.

This orbit was integrated (taking the ratio of the mean motions as 9/5) with the inclination of both the minor planet and Jupiter set to zero. The behaviour of the semi-major axis was virtually unchanged, the oscillations being of the same period and much the same amplitude. The same oscillations also occurred in the eccentricity, although this time they were in opposite phase.

However, there were no large oscillations in the eccentricity. The graph of ψ_2 against ψ_1 was almost exactly the "best-fitting" circle for the corresponding graph in the inclined case (page 190), but the epicycles were entirely absent.

In both integrations, small variations in the movement of the perihelion are indicated by the fact that the small variations in eccentricity appear as pointed cusps in the ψ-curve, the points being inward (that is, the movement of perihelion is slowest when the eccentricity is least). The epicycles on page 190 show that the perihelion also undergoes periodic retrogressions; these disappear when the inclination is neglected. In fact the perihelion moves five times faster when the inclination is neglected, a discrepancy which can hardly be accounted for entirely by the presence or absence of the epicycles.

These results are particularly important in the light of the possibility (page 216) that the existence of the epicycles indicates a "transition" case between commensurable and non-commensurable motion. Thus we may conclude that, while this theoretical approach does not succeed in describing the detailed behaviour of real minor planet orbits, it does give a useful general picture, but only if the inclinations of the orbits are included.

V - Possibilities for Further Research

The most obvious deficiency in the results presented in this thesis is probably the absence of a reliable orbit for (334) Chicago. It is most unfortunate that sequences of observations in two successive seasons both failed to produce acceptable orbits. The computer integration of this orbit produced results that may be regarded with some doubt, both because of the large variations in the parameters, and because this planet can approach rather close to Jupiter; it is therefore particularly desirable to produce a reliable orbit from observations, as a check on these results.

It would also be good to produce an orbit for (1574) Meyer, and additional orbits for the other planets studied here. Most of the orbits that were produced were calculated from only three observations. As described in section II, a computer program was available to "improve" such an orbit by additional observations, but in most cases this program failed, possibly because of the large residuals produced by the original orbit in the additional observations. If a program could be devised based on a less critical technique, it is possible that better orbits could be derived from the existing data.

The positions of the minor planets used for determining the orbits were calculated from the measurements of the plates, by using plate constants. Smart (1958) states that first-order plate constants are adequate in most cases; second-order constants were in fact used here, and it was assumed that these would deal adequately with all the optical and geometrical errors that might arise.

Dodd (1972) uses third-order constants. It was assumed that this was because the telescope used in that research, unlike the James Gregory telescope at St Andrews which was used in this investigation, has a slightly curved focal plane. However, a trial calculation on one of the plates used here gave better results for third-order than for second-order constants: that is, the residuals for all the reference stars were smaller when third-order constants were included. To find if this was true for other plates, and, if so, to apply third-order constants to all the plates, would have meant re-writing the relevant computer program to calculate ten constants instead of six, and time did not permit this. But it would be interesting to pursue this, and the results might be important for all astrometric research carried out on the James Gregory telescope.

The behaviour of the orbits over the long periods of integration has been described on pages 154-212 in fairly general terms. The various oscillations in the orbital parameters are clearly made up of several different terms, each with its own period and amplitude; these have been indicated only approximately in the text. The numerical values given were obtained by visual inspection of the graphs, and by simple arithmetic on selected portions of the printed output.

It would be much more accurate to determine the periods and amplitudes of the various oscillations by calculation on all the output data. For this, it would be desirable to perform Fourier transforms on the different orbital parameters as they vary with time. This should show, for example, which parameters oscillate with exactly the same frequency, and which oscillating term dominates the behaviour of each parameter; it might also be possible to speculate with more confidence on the causes of the various oscillations.

The graphs shown give no direct information on the rate of movement of the perihelion and of the ascending node of each orbit (except that some curves, for example that on page 175, prove that the movement of the perihelion must oscillate in the same way as the eccentricity, in order to give epicycles in the $\psi_1 - \psi_2$ curve). An attempt was made to remedy this by plotting $d\bar{\omega}/dt$; the resultant graphs were of very interesting shape, being periodic but far from sinusoidal, but they suffered greatly from "noise" due to the crudeness of the differentiating technique used (simply taking the differences between every tenth point). If more time were available, it would be interesting to refine the technique and investigate the resultant curves more closely, possibly again by Fourier transforms.

It would also be interesting to compare the results of integrations with and without taking account of the inclinations, for various different planets, to see how large an inclination can be neglected without changing the behaviour of the planets.

The graphs of ψ_2 against ψ_1 and of χ_2 against χ_1 generally approximate to circles (the ψ-curves showing more departures from the circle in the way of cusps and epicycles). The centres and radii of these circles have been estimated, again by inspection of the graphs, and are given only approximately. Schubart (1968) discusses the "proper" elements of each minor planet; the proper eccentricity and the longitude of proper perihelion appear as the radius and angle of the ψ-curve as measured from the centre of the circle instead of from the intersection of the axes. Similarly, the proper inclination and longitude of proper ascending node can be defined as the (arcsine of the) radius and the angle of the χ-curve as measured from the centre of that circle.

Brouwer (1952) lists the proper elements for almost all the minor planets of which the orbits were available at that time, including all those considered here except (1574) Mëyer. The estimated radii of the circles produced here indicate on the whole quite good agreement in the values of proper eccentricity and proper inclination with those of Brouwer. It would be desirable to analyse the curves properly, determining the best-fitting circle by numerical methods, and to compare the results with those of Brouwer. It might also be more meaningful to analyse in detail the behaviour of the proper elements, rather than of the immediate results of the integrations.

The mathematical refinements of the integration results suggested above should not distract attention from the fact that the numerical integrations used here are only an approximation to the physical reality. It would be more relevant to know just how good that approximation is, by more comparison between observed and calculated orbits. To this end, further orbits should be obtained by observation, not just of the planets investigated here, but of all those planets in the vicinity of this commensurability; since there are in fact relatively few of these, attention might perhaps be transferred to some other commensurability, of similar order, where there are more planets. It might then be possible to apply statistical techniques to the results, and to ascertain within what limits of mean motion (or of any other orbital parameter) this method of numerical integration provides an acceptable approximation to the actual movement of the planets.

VI - Appendices

Appendix 1: Plate scale

The conversion factor for standard coordinates between radians and millimetres - the plate-scale - was determined at an early stage of the research. Six known (catalogued) stars on a plate were measured, and the distance d in millimetres between each pair was compared with the separation in their standard coordinates (x_1, y_1) and (x_2, y_2). This gave, for each pair of stars, a value of the plate-scale K:

$$K = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{d} * \frac{180 * 60 * 60}{\pi} \quad ''/\mathrm{mm}$$

The average of all the 15 values of K was 80.19"/mm, and this value was used in all the subsequent calculations.

Appendix 2: Measurements of Pleiades

In order to check the accuracy of the measurement and reduction techniques, it was decided to apply them to a field of stars whose positions and proper motions were well known, namely the Pleiades star-cluster. A plate was taken centred on Alcyone (η Tauri): the observational details are as follows:

date of observation	30 December 1973
coordinates of guide-star	$03^h 45^m 56^s\!.1$ $+24^\circ 01' 33''$
length of exposure	30 minutes
UT of mid-exposure	$21^h 20^m 30^s$

This exposure was taken when the cluster was on the meridian.

The positions, proper motions and magnitudes of the stars investigated were taken from the catalogue given in Leiden Annals volume 19, and the numbers in that catalogue are used throughout this section to identify the stars. The colour indices were taken from Eichhorn (1970).

81 of the stars in the cluster were measured, using the techniques described in section II. For reducing the measurements, 68 of these stars were taken to be reference stars, and the remaining 13 were assumed to be unknown. These 13 were chosen to be evenly scattered over the field of the plate, and also to cover a wide variety of magnitudes and colours; they included Alcyone itself, the guide star (no. 1432).

The reduction process was carried out as described above, in three stages; 22 of the stars were discarded at the final stage. The calculated positions of the 13 "unknown" stars were then compared with their catalogue positions, and the displacements calculated. The average total displacement over the 13 stars was 0.50 seconds of arc.

The residuals for the 46 remaining "reference" stars were then calculated, and were investigated, along with the residuals in the 13 "unknown" stars, to see if any systematic error could be detected. The residuals were found not to correlate either with magnitude m(pg), or with colour B-V. They did not correlate with x, showing that there was no systematic shift of position of the images with time of measurement (since the images were measured in order of increasing x). Nor was there any correlation with y, indicating that the effects of differential refraction (which would change steadily with y, since the plate was taken on the meridian) had been eliminated.

The results of the measurement and reduction are shown in table 4. The residuals in position, $d\alpha$ and $d\delta$, are given for those of the reference stars which were used in the full reduction.

VII - Tables

Table 1: Observational details of all plates used

plate no.	minor planet	guide-star SAO no.	date	mid-exposure LST	exp m.	remarks
12	(909) Ulla	116597	71/12/21-22	$7^h 8^m 40^s$	30	
13	(909) Ulla	116597	71/12/21-22	7 51 40	30	
14	(909) Ulla	116597	71/12/21-22	10 36 50	30	
15	(909) Ulla	116597	71/12/22-23	8 16 10	20	IIaO emulsion
16	(909) Ulla	116597	71/12/23-24	10 50 02	25	IIaO emulsion
17	(334) Chicago	094756	72/ 1/11-12	8 3 54	30	
19	(334) Chicago	094814	72/ 1/14-15	5 4 0	30	
20	(334) Chicago	094756	72/ 1/14-15	5 48 0	30	
22	(334) Chicago	094814	72/ 1/19-20	7 26 0	30	
23	(334) Chicago	094739	72/ 1/19-20	8 0 0 8 37 30	12 13	exposure interrupted: two planet images
24	(909) Ulla	116162	72/ 1/19-20	10 22 0	30	
25	(909) Ulla	116162	72/ 1/19-20	11 8 0	30	
26	(334) Chicago	094723	72/ 1/20-21	6 54 20	30	compared with no.23
27	(909) Ulla	116162	72/ 1/20-21	7 39 20	30	

29	(909) Ulla	097224	72/ 2/10-11	$8^h53^m0^s$	30^m	
31	(909) Ulla	097224	72/ 2/11-12	7 13 0	30	
33	(909) Ulla	097224	72/ 2/12-13	6 25 0 6 58 0	10 20	exposure interrupted: two planet images
34	(909) Ulla	097224	72/ 2/13-14	6 1 0	30	
35	(909) Ulla	097224	72/ 2/14-15	5 31 0	30	
36	(1574) Meyer	074166	72/ 9/ 7- 8	23 13 30	20	
38	(1574) Meyer	074127	72/ 9/ 7- 8	0 27 30	20	compared with nos.36 & 40
40	(1574) Meyer	074166	72/ 9/ 9-10	22 27 30	30	
42	(909) Ulla	119209	73/ 2/ 6- 7	9 34 0	20	
43	(334) Chicago	098521	73/ 2/ 6- 7	10 17 0	20	compared with no.45 (not used)
44	(909) Ulla	119209	73/ 2/ 6- 7	10 51 0	20	
46	(909) Ulla	119224	73/ 2/ 9-10	9 51 0	6	terminated by cloud: compared with no.55 (not used)
47	(334) Chicago	098389	73/ 2/26-27	14 6 0	20	
49	(334) Chicago	098389	73/ 2/27-28	14 0 0	20	
50	(909) Ulla	099910	73/ 2/27-28	14 35 0	20	compared with nos.52 & 54
51	(334) Chicago	098358	73/ 3/ 4- 5	8 56 0	20	
52	(909) Ulla	099852	73/ 3/ 4- 5	9 36 0	20	

53	(334) Chicago	098358	73/ 3/ 7- 8	$13^h 39^m 0^s$	20^m	
54	(909) Ulla	099852	73/ 3/ 7- 8	14 10 0	20	
56	(909) Ulla	099774	73/ 3/24-25	10 9 0	10	
57	(909) Ulla	099774	73/ 3/24-25	10 29 0	10	
58	(909) Ulla	099774	73/ 3/25-26	10 32 30 10 45 30	7 3	exposure interrupted by cloud: planet image blurred exposure time taken as $10^h 36^m 24^s$
59	(909) Ulla	099688	73/ 3/30-31	14 44 0	10	
60	(909) Ulla	099688	73/ 3/30-31	15 5 0	10	
61	(909) Ulla	099702	73/ 4/ 4- 5	15 5 0	10	
62	(909) Ulla	099702	73/ 4/ 4- 5	15 26 0	10	
81	(414) Liriope	094218	73/10/22-23	23 46 30	15	
82	(414) Liriope	094218	73/10/22-23	0 17 0	15	
83	(414) Liriope	094218	73/10/26-27	1 10 0	15	
85	(414) Liriope	094252	73/10/26-27	2 12 30	15	
86	(414) Liriope	094252	73/10/28-29	4 24 30	15	
88	(414) Liriope	094063	73/11/22-23	8 31 0	15	
90	(87) Sylvia	078580	73/11/22-23	9 29 0	15	compared with Palomar print; found in star-cluster NGC 2266
97	(414) Liriope	094044	73/11/27-28	4 41 30	15	compared with no.88

98	(107) Camilla	114515	73/11/27-28	$5^h 10^m 30^s$	15^m	compared with Palomar print
99	(87) Sylvia	078524	73/11/27-28	5 37 30	15	compared with no.90
100	(107) Camilla	095870	73/12/22-23	10 5 20	15	
101	(414) Liriope	093869	73/12/23-24	6 10 30	15	compared with nos.103 & 106 (not used)
102	(87) Sylvia	078233	73/12/23-24	6 44 0	15	
104	(701) Camilla	095870	73/12/25-26	6 48 0	15	
105	(87) Sylvia	078233	73/12/25-26	7 21 0	15	
107	(701) Camilla	095763	73/12/29-30	7 34 20	15	compared with nos.100 & 104
108	(87) Sylvia	078165	73/12/29-30	8 1 30	15	compared with nos.102 & 105
110	(414) Liriope	093772	74/ 1/15-16	3 13 50	10	compared with no.113 (not used)
111	(107) Camilla	113699	74/ 1/15-16	3 45 20	15	
112	(87) Sylvia	077830	74/ 1/15-16	4 8 45	15	
114	(107) Camilla	095453	74/ 1/17-18	2 50 0	15	compared with no.111
115	(87) Sylvia	077830	74/ 1/17-18	3 16 50	15	

Table 2: Reference stars used

SAO no.	R.A.	μ_α	dec.	μ_δ	m	minor planet	on plates
058638	$5^h 56^m 24.273^s$	-0.0003^s	$+30° 0' 10.56''$	$-0.012''$	8.5^m	(87) Sylvia	112,115
074085	0 29 59.955	-0.0005	+21 44 20.08	-0.012	8.7	(1574) Meyer	38
074092	0 30 25.446	-0.0006	+21 9 3.91	-0.006	8.8	(1574) Meyer	38
074095	0 30 51.812	0.0014	+20 35 20.43	0.004	9.1	(1574) Meyer	38
074106	0 31 40.103	-0.0007	+22 10 18.03	0.001	8.7	(1574) Meyer	38
074118	0 32 29.304	-0.0029	+21 55 0.01	-0.014	8.8	(1574) Meyer	36,38,40
074127	0 33 10.524	-0.0009	+21 29 54.74	-0.010	8.2	(1574) Meyer	36,38(c),40
074130	0 33 14.293	0.0012	+20 40 47.48	0.002	8.5	(1574) Meyer	36,38,40
074131	0 33 15.703	-0.0063	+21 3 11.88	-0.053	8.3	(1574) Meyer	36,38,40
074137	0 33 41.204	0.0004	+21 18 17.08	-0.005	8.5	(1574) Meyer	36,38,40
074146	0 34 17.800	0.0011	+22 1 43.80	-0.004	8.7	(1574) Meyer	36,38,40
074149	0 34 41.322	0.0001	+21 20 12.41	-0.010	9.1	(1574) Meyer	36,40
074150	0 34 51.557	0.0006	+21 38 41.16	0.012	8.5	(1574) Meyer	36,38,40

074152	0h 34m 58.104s	-0.0016s	+22° 23' 59.05"	0.007"	8.7m	(1574) Meyer	36,38,40
074153	0 35 1.839	0.0024	+21 7 40.88	0.000	8.8	(1574) Meyer	36,38,40
074154	0 35 2.651	0.0042	+20 49 28.53	-0.047	7.0	(1574) Meyer	36,38,40
074160	0 35 22.616	-0.0012	+22 13 16.75	0.027	8.6	(1574) Meyer	36,40
074166	0 36 7.499	-0.0018	+21 28 12.43	-0.006	7.8	(1574) Meyer	36(c),38,40(c)
074171	0 36 24.059	0.0069	+20 31 54.56	-0.002	9.0	(1574) Meyer	36,40
074195	0 38 20.951	-0.0011	+20 44 46.79	-0.014	7.3	(1574) Meyer	36,40
077347	5 35 40.871	0.0000	+20 14 13.08	-0.005	9.1	(334) Chicago	23,26
077362	5 36 47.298	-0.0001	+20 24 10.17	-0.002	9.1	(334) Chicago	23,26
077395	5 38 13.666	0.0002	+20 2 15.87	-0.010	8.0	(334) Chicago	23,26
077734	5 52 54.899	0.0006	+29 0 46.26	-0.041	9.1	(87) Sylvia	112,115
077768	5 54 31.382	0.0009	+28 17 10.30	0.008	8.4	(87) Sylvia	112,115
077771	5 54 38.585	0.0005	+29 48 28.35	0.015	8.8	(87) Sylvia	112,115
077779	5 55 3.171	0.0012	+29 24 27.96	-0.009	8.7	(87) Sylvia	112,115
077818	5 56 55.485	0.0004	+28 7 26.84	-0.018	7.0	(87) Sylvia	112,115
077820	5 57 0.195	-0.0014	+29 25 4.38	-0.024	8.8	(87) Sylvia	112(f),115

077830	$5^h 57^m 26^s.100$	$0^s.0001$	$+29^\circ 0' 16''.05$	$-0''.019$	$7^m.4$	(87) Sylvia	112(c),115(c)
077834	5 57 30.539	-0.0002	+29 16 17.09	-0.015	9.0	(87) Sylvia	115(f)
077843	5 58 7.656	0.0000	+28 48 18.42	-0.041	8.8	(87) Sylvia	112,115
077844	5 58 8.328	-0.0012	+29 26 34.97	-0.016	8.4	(87) Sylvia	115(f)
077847	5 58 13.614	0.0006	+29 34 38.65	-0.019	8.4	(87) Sylvia	112,115
077865	5 58 57.131	0.0012	+29 59 54.69	-0.024	9.1	(87) Sylvia	112,115
077866	5 59 0.895	0.0011	+28 8 32.96	0.003	9.0	(87) Sylvia	112,115
077899	6 0 24.638	-0.0003	+29 20 10.19	0.005	9.2	(87) Sylvia	112(f)
077908	6 0 50.291	0.0010	+28 38 57.41	-0.007	8.7	(87) Sylvia	112,115(f)
077914	6 1 3.990	-0.0008	+29 31 20.94	-0.026	8.7	(87) Sylvia	112,115
077917	6 1 6.508	-0.0008	+28 18 15.51	0.003	8.3	(87) Sylvia	112(f),115
077919	6 1 12.689	-0.0006	+28 59 17.04	0.009	8.5	(87) Sylvia	112(f),115
077940	6 2 19.990	-0.0098	+29 8 2.30	-0.010	9.3	(87) Sylvia	112
078102	6 9 40.365	0.0001	+29 24 4.36	-0.019	8.9	(87) Sylvia	108
078112	6 10 5.779	0.0004	+29 2 55.01	-0.011	8.2	(87) Sylvia	108
078117	6 10 29.490	-0.0032	+28 22 14.20	-0.043	9.2	(87) Sylvia	108
078123	6 11 7.455	-0.0024	0+28 43 13.02	-0.042	9.0	(87) Sylvia	108

078126	6h 11m 13.824s	0.0011s	+28° 2′ 38.59″	-0.022″	8.7m	(87) Sylvia	108
078138	6 11 58.279	-0.0006	+29 17 46.98	-0.031	8.3	(87) Sylvia	108
078147	6 12 19.462	-0.0005	+28 59 11.30	-0.048	8.7	(87) Sylvia	108
078149	6 12 45.706	-0.0003	+27 52 46.01	-0.025	8.0	(87) Sylvia	108
078159	6 13 4.025	-0.0067	+28 13 3.17	-0.011	9.1	(87) Sylvia	105(s),108
078163	6 13 9.133	-0.0028	+28 17 12.24	0.022	8.8	(87) Sylvia	102,105(f)
078165	6 13 11.734	-0.0004	+28 52 11.98	-0.027	7.3	(87) Sylvia	102,105,108(c)
078169	6 13 16.012	-0.0023	+29 42 51.89	-0.028	9.0	(87) Sylvia	108
078175	6 13 53.727	-0.0020	+29 33 40.42	-0.050	8.5	(87) Sylvia	108
078177	6 13 55.922	-0.0011	+27 52 10.15	-0.041	9.0	(87) Sylvia	102(f),105
078185	6 14 32.936	0.0007	+28 5 25.03	-0.019	9.1	(87) Sylvia	102(f),105(f)
078186	6 14 34.656	0.0000	+29 14 22.06	-0.014	7.7	(87) Sylvia	102,105(f),108
078191	6 14 50.718	0.0011	+28 1 36.57	-0.013	7.4	(87) Sylvia	102(s),105(s),108
078206	6 15 38.007	0.0000	+28 4 34.84	-0.008	8.0	(87) Sylvia	102(f),105(f)
078213	6 16 2.099	0.0028	+29 24 32.27	-0.014	8.9	(87) Sylvia	102,105
078214	6 16 5.569	-0.0014	+28 27 40.01	-0.014	9.3	(87) Sylvia	102(f),105(f)
078220	6 16 12.871	0.0010	+29 27 50.98	-0.023	9.3	(87) Sylvia	108
078233	6 16 49.137	0.0005	+28 26 58.18	-0.037	7.2	(87) Sylvia	102(c),105(c),108

	h m s	s	° ' "	"	m		
078235	6 16 58.185	-0.0027	+27 23 16.55	-0.001	9.0	(87) Sylvia	102(f),105
078237	6 17 3.013	0.0002	+29 31 28.19	-0.014	8.8	(87) Sylvia	102(f)
078240	6 17 12.138	-0.0009	+28 5 59.88	-0.005	8.6	(87) Sylvia	102,105
078242	6 17 17.403	-0.0013	+27 38 57.45	-0.026	8.7	(87) Sylvia	102,105
078244	6 17 21.878	0.0009	+28 42 40.02	-0.008	8.7	(87) Sylvia	102,105
078253	6 17 37.183	0.0001	+29 35 43.35	0.009	9.0	(87) Sylvia	102(s),105(s)
078254	6 17 38.461	0.0001	+28 59 19.06	-0.020	9.0	(87) Sylvia	102,105,108
078257	6 17 52.941	-0.0029	+29 23 42.84	0.013	6.9	(87) Sylvia	102(f)
078259	6 18 0.479	0.0028	+29 33 55.66	-0.040	6.3	(87) Sylvia	105(f)
078262	6 18 7.609	-0.0037	+29 16 35.79	0.000	9.0	(87) Sylvia	102,105
078271	6 18 45.780	0.0004	+27 32 54.58	-0.002	9.1	(87) Sylvia	102,105
078282	6 19 17.599	-0.0005	+28 56 15.04	-0.007	7.9	(87) Sylvia	102,105(f)
078283	6 19 17.778	-0.0001	+28 8 54.98	-0.008	8.8	(87) Sylvia	102(f),105(f)
078290	6 19 37.798	-0.0013	+29 27 8.14	0.017	9.1	(87) Sylvia	102(s),105(s)
078291	6 19 38.737	-0.0019	+28 0 45.07	-0.002	7.5	(87) Sylvia	102(s),105(s)
078294	6 19 50.226	0.0006	+28 38 50.57	-0.026	7.7	(87) Sylvia	105(s)
078299	6 19 57.060	-0.0023	+27 45 26.94	0.005	7.9	(87) Sylvia	102(s),105(f)
078302	6 20 7.453	0.0002	+28 36 11.91	-0.022	8.4	(87) Sylvia	102,105(f)

078318	6 20 41.613	0.0008	+29 0 6.84	-0.001	8.9	(87) Sylvia	105(s)
078326	6 21 16.165	0.0031	+28 32 12.75	-0.060	9.5	(87) Sylvia	102(f),105(f)
078470	6 29 12.915	0.0014	+28 49 29.86	-0.024	9.1	(87) Sylvia	99
078478	6 29 46.133	0.0006	+28 22 18.23	0.006	9.0	(87) Sylvia	99
078480	6 29 53.357	-0.0001	+27 51 48.94	-0.018	7.5	(87) Sylvia	99
078497	6 30 39.743	-0.0005	+28 24 24.28	0.011	9.4	(87) Sylvia	99
078504	6 30 56.608	-0.0035	+27 35 38.76	-0.010	9.3	(87) Sylvia	99(f)
078506	6 30 59.888	0.0001	+27 55 24.14	0.003	9.0	(87) Sylvia	99
078509	6 31 9.116	-0.0002	+27 39 8.37	-0.036	9.3	(87) Sylvia	99(s)
078512	6 31 26.373	0.0064	+28 52 18.85	-0.126	9.0	(87) Sylvia	99
078524	6 32 3.125	0.0003	+28 3 47.31	-0.018	5.0	(87) Sylvia	99(c)
078527	6 32 13.932	-0.0004	+27 35 31.60	0.002	9.0	(87) Sylvia	90,99
078531	6 32 33.814	-0.0022	+27 43 26.44	-0.034	8.8	(87) Sylvia	90,99
078532	6 32 34.028	0.0006	+28 16 42.06	-0.009	9.1	(87) Sylvia	90,99
078533	6 32 35.536	-0.0004	+28 4 48.73	0.021	9.3	(87) Sylvia	90,99
078541	6 33 23.811	0.0038	+27 56 0.78	0.019	9.4	(87) Sylvia	90,99
078547	6 33 44.952	-0.0002	+26 58 56.73	-0.015	9.2	(87) Sylvia	90,99

078552	6 34 10.530	-0.0026	+27 19 6.93	-0.010	9.1	(87) Sylvia	90
078553	6 34 12.710	-0.0032	+28 1 9.83	0.006	8.9	(87) Sylvia	90
078554	6 34 14.461	0.0015	+28 58 28.53	-0.013	9.0	(87) Sylvia	99
078576	6 35 28.316	-0.0001	+28 57 1.44	-0.009	8.0	(87) Sylvia	90
078577	6 35 28.642	-0.0013	+28 43 3.11	0.004	9.0	(87) Sylvia	99
078580	6 35 45.187	-0.0003	+27 50 58.98	-0.037	7.4	(87) Sylvia	90(c)
078595	6 36 25.756	-0.0032	+28 24 56.33	-0.012	8.7	(87) Sylvia	90,99
078608	6 36 59.342	-0.0007	+28 56 31.88	-0.015	8.9	(87) Sylvia	90
078614	6 37 11.453	0.0021	+28 43 51.10	0.007	9.2	(87) Sylvia	90
078616	6 37 13.825	0.0015	+28 10 26.97	0.009	9.1	(87) Sylvia	90,99
078618	6 37 16.114	0.0007	+28 26 57.99	-0.031	9.0	(87) Sylvia	90
078640	6 38 23.090	0.0009	+26 49 15.07	-0.008	9.2	(87) Sylvia	90
078641	6 38 31.352	0.0010	+27 26 37.74	-0.008	8.7	(87) Sylvia	90,99
078642	6 38 31.953	0.0013	+28 21 8.44	-0.006	9.2	(87) Sylvia	90
078648	6 38 54.675	0.0004	+27 12 18.02	0.004	8.5	(87) Sylvia	90
078660	6 39 26.369	0.0009	+27 29 5.13	-0.009	8.9	(87) Sylvia	90
093738	4 0 15.913	-0.0005	+13 5 56.28	-0.020	8.7	(414) Liriope	110

093741	4 0 37.680	0.0007	+13 48 20.77	-0.018	8.8	(414) Liriope	110
093743	4 0 44.923	-0.0004	+12 45 42.93	-0.013	8.7	(414) Liriope	110
093750	4 2 23.691	-0.0001	+13 9 41.48	-0.020	8.0	(414) Liriope	110
093754	4 2 42.540	0.0016	+14 9 7.04	-0.035	7.4	(414) Liriope	110
093764	4 3 44.280	0.0005	+14 14 11.35	-0.080	8.6	(414) Liriope	110
093767	4 3 53.137	-0.0015	+13 1 25.92	-0.026	8.8	(414) Liriope	110
093768	4 4 3.884	0.0005	+13 35 59.76	-0.094	8.7	(414) Liriope	110
093769	4 4 22.539	0.0003	+13 40 45.96	-0.030	8.5	(414) Liriope	110
093772	4 4 31.046	-0.0015	+13 24 19.68	-0.004	8.2	(414) Liriope	110(c)
093773	4 4 31.647	-0.0010	+14 36 24.31	-0.033	9.2	(414) Liriope	110
093779	4 5 30.336	0.0057	+13 23 31.26	-0.011	8.8	(414) Liriope	110
093782	4 6 1.599	0.0006	+12 53 28.68	-0.025	9.0	(414) Liriope	110
093790	4 6 38.357	0.0002	+13 16 16.99	0.000	8.9	(414) Liriope	110
093804	4 8 10.494	-0.0014	+14 8 56.57	-0.030	8.5	(414) Liriope	110
093839	4 13 58.161	0.0019	+12 50 6.64	-0.018	9.0	(414) Liriope	101
093847	4 14 54.419	0.0064	+13 42 51.29	0.046	7.0	(414) Liriope	101
093852	4 15 35.886	-0.0008	+12 27 24.11	-0.013	8.8	(414) Liriope	101

093857	4h 16m 4.609s	-0.0001s	+12° 48′ 16.77″	-0.028″	8.6m	(414) Liriope	101
093858	4 16 6.570	0.0004	+14 4 34.01	-0.012	8.8	(414) Liriope	101
093859	4 16 14.732	0.0005	+12 8 53.07	-0.067	8.5	(414) Liriope	101
093861	4 16 20.678	0.0015	+13 36 3.66	0.013	7.7	(414) Liriope	101
093864	4 16 42.142	-0.0001	+11 50 47.50	0.010	7.9	(414) Liriope	101
093869	4 16 59.646	-0.0023	+12 57 59.53	-0.009	8.1	(414) Liriope	101(c)
093872	4 17 8.522	0.0079	+13 54 58.43	-0.019	5.6	(414) Liriope	101
093875	4 17 37.697	-0.0021	+12 16 41.88	-0.003	9.0	(414) Liriope	101
093878	4 18 3.770	0.0077	+13 44 46.97	-0.023	6.1	(414) Liriope	101
093889	4 18 57.055	0.0006	+13 28 3.84	0.003	8.0	(414) Liriope	101
093891	4 19 13.748	0.0026	+12 23 28.89	0.081	8.8	(414) Liriope	101
093892	4 19 14.239	0.0077	+13 57 37.91	-0.024	5.8	(414) Liriope	101
093908	4 21 16.576	0.0036	+12 51 29.88	-0.027	8.5	(414) Liriope	101
094013	4 31 38.182	0.0017	+12 38 8.35	-0.335	9.0	(414) Liriope	97
094023	4 32 46.749	0.0026	+11 59 55.80	-0.022	8.5	(414) Liriope	88,97
094044	4 35 21.558	0.0067	+12 24 43.81	-0.010	4.3	(414) Liriope	97(c)
094050	4 36 9.097	-0.0001	+11 53 57.57	-0.022	8.4	(414) Liriope	88,97

	h m s	s	° ′ ″	″	m		
094052	4 36 18.902	0.0013	+12 25 40.05	-0.006	8.8	(414) Liriope	88,97
094057	4 36 47.417	0.0159	+13 1 40.16	-0.147	8.8	(414) Liriope	88,97
094058	4 36 50.122	0.0040	+13 34 29.66	-0.078	9.1	(414) Liriope	97
094062	4 37 11.198	0.0045	+13 23 21.88	-0.087	8.8	(414) Liriope	97
094063	4 37 16.174	-0.0002	+12 6 3.97	-0.007	5.4	(414) Liriope	88(c)
094066	4 38 20.886	0.0016	+12 29 29.80	-0.012	8.9	(414) Liriope	88,97
094077	4 39 49.999	-0.0000	+12 13 2.81	0.014	8.2	(414) Liriope	88,97
094086	4 40 34.134	0.0012	+12 21 11.72	-0.001	9.0	(414) Liriope	88
094087	4 40 39.947	0.0016	+12 56 35.94	-0.044	8.8	(414) Liriope	88
094171	4 49 0.607	0.0076	+13 34 19.42	-0.030	6.7	(414) Liriope	81,82,83
094176	4 49 42.058	-0.0001	+14 10 8.31	-0.057	5.2	(414) Liriope	81,82,83
094190	4 51 40.785	0.0068	+12 22 17.16	-0.039	8.6	(414) Liriope	81,82,83
094193	4 51 55.743	0.0081	+12 16 21.80	-0.055	7.1	(414) Liriope	81,82,83
094202	4 52 9.851	0.0007	+14 30 51.09	-0.010	7.7	(414) Liriope	81,82,83
094213	4 53 2.909	0.0002	+13 33 8.37	-0.004	8.1	(414) Liriope	81,82,83
094216	4 53 21.029	0.0066	+13 53 53.18	-0.086	8.8	(414) Liriope	81,82,83
094218	4 53 33.423	-0.0051	+13 26 13.64	-0.046	4.3	(414) Liriope	81(c),82(c),83(c), 85(f),86

094226	4 54 28.522	-0.0012	+12 11 13.67	0.010	9.0	(414) Liriope	85,86
094230	4 54 51.885	0.0016	+13 51 23.03	-0.031	9.1	(414) Liriope	81,82,83
094231	4 54 59.576	0.0058	+13 55 34.20	-0.016	8.7	(414) Liriope	81,82,83
094234	4 55 29.489	0.0021	+13 52 42.01	-0.082	8.6	(414) Liriope	81,82,83
094235	4 55 36.490	-0.0009	+12 42 21.11	-0.009	8.7	(414) Liriope	81,82,83,85(f),86(f)
094238	4 56 2.866	0.0021	+11 52 54.48	-0.010	8.7	(414) Liriope	85,86
094241	4 56 9.413	-0.0007	+12 20 25.38	-0.018	8.5	(414) Liriope	85,86
094242	4 56 11.369	-0.0001	+12 53 8.11	0.011	8.9	(414) Liriope	81,82,83,85,86
094244	4 56 12.768	0.0001	+14 28 7.11	-0.019	9.0	(414) Liriope	81,82,83
094245	4 56 19.918	-0.0010	+12 51 16.31	-0.019	9.0	(414) Liriope	81,82,83,85
094246	4 56 43.378	0.0003	+12 45 6.26	-0.007	8.5	(414) Liriope	81,82,83,85,86
094250	4 57 25.236	0.0005	+12 20 46.97	-0.013	8.9	(414) Liriope	85,86
094252	4 57 40.831	-0.0004	+12 30 8.69	-0.008	8.4	(414) Liriope	81,82(s),85(c),86(c)
094254	4 57 49.236	-0.0008	+13 45 8.68	-0.023	9.0	(414) Liriope	82,83
094255	4 57 53.619	0.0005	+12 31 52.59	-0.017	8.3	(414) Liriope	82(f),85,86
094258	4 57 58.266	0.0001	+12 56 45.87	0.002	9.0	(414) Liriope	81,82,83,85,86
094261	4 58 12.906	0.0004	+13 13 10.75	-0.009	9.0	(414) Liriope	81,82,83,85,86
094268	4 58 29.830	-0.0006	+13 1 34.49	-0.015	8.8	(414) Liriope	81,82,85,86

	h m s	s	° ′ ″	″	m		
094272	4 58 48.776	0.0032	+12 16 14.63	-0.107	7.9	(414) Liriope	85,86
094281	4 59 48.055	-0.0010	+12 59 35.79	-0.015	8.9	(414) Liriope	85,86
094285	5 0 14.171	0.0004	+11 59 19.18	0.054	7.1	(414) Liriope	85,86
094286	5 0 36.508	-0.0023	+12 36 16.83	-0.010	8.4	(414) Liriope	85,86
094292	5 1 49.905	0.0002	+12 37 49.98	-0.012	8.5	(414) Liriope	85
094684	5 33 5.313	-0.0000	+19 31 6.80	-0.023	7.6	(334) Chicago	23,26
094685	5 33 8.864	-0.0002	+19 34 7.57	-0.020	8.8	(334) Chicago	23,26
094688	5 33 15.810	-0.0012	+19 37 54.21	0.004	8.9	(334) Chicago	23,26
094710	5 34 36.808	-0.0004	+19 45 2.54	0.002	8.5	(334) Chicago	23,26
094723	5 36 10.500	0.0007	+19 44 29.15	-0.003	8.0	(334) Chicago	23,26(c)
094725	5 36 14.475	0.0002	+18 54 29.61	-0.001	9.0	(334) Chicago	20,23,26
094735	5 36 57.657	-0.0007	+19 21 41.51	-0.013	9.0	(334) Chicago	17,20,23,26
094739	5 37 12.066	0.0002	+19 39 15.56	-0.008	7.8	(334) Chicago	23(c),26
094751	5 37 49.139	-0.0005	+19 6 3.32	-0.027	8.8	(334) Chicago	17,19,20,22,23,26
094756	5 38 12.773	-0.0009	+18 35 59.77	-0.022	8.7	(334) Chicago	17(c),19(s),20(c),22
094763	5 38 37.809	0.0007	+19 11 46.86	-0.003	8.2	(334) Chicago	17,19,20,22,23,26
094767	5 38 47.046	-0.0007	+19 19 59.21	-0.016	8.2	(334) Chicago	17,19,20,22,23,26

094770	5h 38m 49.048s	0.0001s	+19° 25′ 3.35″	-0.033″	8.5m	(334) Chicago	17,19,20,22,23,26
094773	5 38 54.402	0.0001	+19 38 29.43	-0.003	8.9	(334) Chicago	19,22,23,26
094776	5 39 2.743	-0.0005	+19 39 30.48	-0.015	8.9	(334) Chicago	19,22,23,26
094779	5 39 4.001	0.0008	+18 31 2.00	-0.013	9.1	(334) Chicago	17,19(s),20,22
094782	5 39 18.107	0.0004	+19 37 12.29	-0.008	8.9	(334) Chicago	19,22,23,26
094784	5 39 28.004	0.0024	+18 39 53.26	-0.037	8.1	(334) Chicago	17,19,20,22
094786	5 39 31.270	-0.0012	+18 38 19.00	-0.004	9.0	(334) Chicago	17(f),19,20,22
094787	5 39 32.106	0.0004	+18 57 52.97	-0.032	7.5	(334) Chicago	17,19,20,22(s)
094791	5 39 51.885	-0.0011	+19 49 29.65	-0.016	9.1	(334) Chicago	19,22
094793	5 39 57.076	-0.0003	+18 57 29.26	-0.006	6.7	(334) Chicago	17(f),19,20,22
094796	5 40 11.337	-0.0009	+18 29 51.45	-0.017	8.4	(334) Chicago	17,19(s),20,22
094798	5 40 18.333	-0.0001	+18 48 47.22	-0.037	9.0	(334) Chicago	17,19,20,22
094807	5 40 47.987	-0.0000	+19 2 50.49	-0.008	9.1	(334) Chicago	17,19,20,22
094809	5 40 53.287	-0.0002	+18 37 35.34	0.001	9.0	(334) Chicago	17,19,20,22
094812	5 41 10.729	-0.0002	+19 50 27.68	-0.018	9.1	(334) Chicago	19,22
094814	5 41 20.072	-0.0013	+18 48 48.90	-0.008	7.5	(334) Chicago	17,19(c),20,22(c)
094815	5 41 26.123	0.0013	+18 30 22.04	-0.012	8.2	(334) Chicago	17,19(s)
094817	5 41 40.028	0.0001	+19 36 54.81	0.013	9.0	(334) Chicago	19,22

094822	$5^h 41^m 56.^s958$	$0.^s0017$	$+18° 43' 4.''21$	$-0.''008$	$8.^m9$	(334) Chicago	17,19,20
094830	5 42 15.243	-0.0002	+18 41 4.77	0.003	6.9	(334) Chicago	17(f),19(s)
094833	5 42 26.947	-0.0000	+18 42 28.43	-0.018	9.2	(334) Chicago	19(s)
094839	5 42 52.809	-0.0001	+18 46 24.65	-0.013	7.6	(334) Chicago	19(s)
095348	6 8 28.159	-0.0029	+10 0 57.18	-0.009	9.1	(107) Camilla	114
095363	6 9 8.725	-0.0019	+11 2 18.98	-0.010	8.9	(107) Camilla	114(f)
095376	6 9 35.152	-0.0005	+10 18 58.79	-0.006	8.9	(107) Camilla	114(f)
095388	6 10 10.863	-0.0003	+10 46 51.04	0.009	9.0	(107) Camilla	114
095420	6 11 33.826	-0.0011	+10 9 15.04	-0.011	8.5	(107) Camilla	111,114
095428	6 11 48.835	-0.0009	+11 5 37.38	-0.042	8.8	(107) Camilla	114
095441	6 12 8.274	0.0010	+11 40 8.33	-0.003	8.3	(107) Camilla	114
095453	6 12 43.091	0.0011	+10 35 46.17	-0.021	8.0	(107) Camilla	114(c)
095458	6 13 4.752	-0.0009	+11 13 6.74	-0.013	9.1	(107) Camilla	114(f)
095460	6 13 10.663	-0.0020	+10 17 59.54	-0.037	6.8	(107) Camilla	111,114
095474	6 13 34.559	0.0002	+11 28 48.91	-0.010	8.7	(107) Camilla	114
095490	6 14 33.349	-0.0008	+10 56 20.22	-0.004	8.7	(107) Camilla	114
095517	6 16 2.577	-0.0004	+11 5 12.02	-0.022	8.5	(107) Camilla	114

095522	$6^h 16^m 13^s.056$	$0^s.0003$	$+10° 20' 50''.93$	$-0''.027$	$8^m.9$	(107) Camilla	111,114
095528	6 16 20.478	-0.0005	+10 38 26.06	-0.002	8.7	(107) Camilla	114(f)
095587	6 19 19.107	-0.0009	+10 2 57.25	-0.006	8.8	(107) Camilla	111
095699	6 24 31.932	-0.0005	+10 21 11.77	-0.011	8.9	(107) Camilla	107
095711	6 25 13.736	-0.0015	+10 2 12.71	-0.033	8.5	(107) Camilla	107(f)
095723	6 25 39.770	-0.0007	+10 59 6.24	-0.003	8.8	(107) Camilla	107
095727	6 26 3.151	-0.0019	+10 16 19.78	-0.012	8.7	(107) Camilla	107
095739	6 26 35.659	0.0004	+10 55 50.92	0.002	8.8	(107) Camilla	107(f)
095746	6 26 55.522	-0.0006	+10 41 59.14	0.023	8.1	(107) Camilla	107
095761	6 27 32.553	-0.0005	+10 28 21.69	-0.008	8.8	(107) Camilla	107
095763	6 27 36.640	-0.0001	+10 9 52.23	-0.025	8.2	(107) Camilla	107(c)
095769	6 28 26.871	-0.0019	+10 9 15.07	-0.006	8.6	(107) Camilla	107
095785	6 28 54.167	-0.0009	+11 8 20.06	-0.011	8.8	(107) Camilla	107
095786	6 28 57.467	-0.0016	+10 22 33.56	-0.006	8.9	(107) Camilla	100(f),104(f)
095807	6 29 41.864	-0.0017	+10 0 47.40	-0.031	9.2	(107) Camilla	100(f),104(f),107
095813	6 29 53.372	0.0001	+10 27 40.69	-0.006	8.6	(107) Camilla	107
095816	6 30 2.941	0.0004	+10 12 15.63	-0.005	8.0	(107) Camilla	100(f),104(f)

	h m s	s	° ′ ″	″	m		
095823	6 30 19.367	-0.0014	+10 21 38.17	-0.011	8.8	(107) Camilla	104(s)
095835	6 30 51.086	-0.0009	+10 34 10.98	-0.019	8.9	(107) Camilla	100,104(f)
095836	6 30 52.604	-0.0020	+10 3 37.96	-0.013	8.6	(107) Camilla	100(f),104(f)
095840	6 31 3.874	0.0009	+10 13 6.85	-0.017	8.6	(107) Camilla	107
095870	6 32 32.164	-0.0010	+10 1 46.71	0.002	6.1	(107) Camilla	100(c),104(c)
095874	6 32 41.084	-0.0014	+10 53 2.70	-0.021	8.7	(107) Camilla	100,104
095893	6 33 23.949	-0.0007	+10 19 36.83	-0.004	7.9	(107) Camilla	100(f),104
095896	6 33 37.799	-0.0011	+10 34 24.34	-0.028	9.5	(107) Camilla	100,104(f)
095914	6 34 50.614	-0.0027	+10 53 50.73	-0.040	6.6	(107) Camilla	100,104
095935	6 35 47.220	-0.0010	+10 38 41.46	-0.016	8.9	(107) Camilla	100,104(f)
095957	6 36 49.351	-0.0005	+10 17 12.38	-0.008	8.5	(107) Camilla	100,104
096127	6 45 25.235	-0.0011	+10 14 45.30	-0.011	8.8	(107) Camilla	98
096174	6 47 22.389	-0.0012	+10 5 22.47	-0.001	8.6	(107) Camilla	98
096224	6 50 2.050	-0.0009	+10 4 54.05	-0.004	8.8	(107) Camilla	98
096235	6 50 21.427	0.0004	+10 32 27.63	-0.045	8.5	(107) Camilla	98
096255	6 51 36.120	-0.0006	+10 6 36.53	-0.031	8.6	(107) Camilla	98

097151	7h 38m 59.759s	-0.0000s	+11° 0′ 11.71″	-0.006″	8.8m	(909) Ulla	29,31,33,34,35
097155	7 39 10.552	-0.0006	+10 29 31.53	-0.011	9.0	(909) Ulla	29,31,33,34,35
097159	7 39 15.922	0.0013	+10 48 38.45	-0.032	8.9	(909) Ulla	29,31,33,34,35
097175	7 40 4.339	0.0023	+11 26 49.95	-0.075	8.7	(909) Ulla	29,31,33,34,35
097176	7 40 5.622	-0.0006	+10 41 2.83	-0.015	8.6	(909) Ulla	29,31,33,34,35
097179	7 40 19.982	-0.0013	+11 49 53.45	-0.024	8.9	(909) Ulla	31,33,34,35
097181	7 40 25.426	-0.0000	+11 22 20.64	0.031	9.0	(909) Ulla	29,31,33,34,35
097184	7 40 35.324	-0.0002	+11 19 58.78	-0.011	8.9	(909) Ulla	29,31,33,34,35
097186	7 40 40.549	-0.0005	+12 0 21.94	0.001	8.7	(909) Ulla	33(s),34,35
097204	7 41 41.499	-0.0007	+10 58 36.47	0.003	8.9	(909) Ulla	29,31,33,34,35
097213	7 42 29.012	-0.0010	+10 30 33.04	0.003	9.1	(909) Ulla	29,31,33,34,35
097217	7 42 47.983	-0.0007	+11 51 57.40	-0.006	8.3	(909) Ulla	33,34,35
097219	7 43 1.517	0.0005	+10 31 8.62	-0.034	7.7	(909) Ulla	29,31,33,34,35
097224	7 43 31.103	-0.0019	+10 53 29.87	-0.024	5.3	(909) Ulla	(*)29,31,33,34,35
097225	7 43 46.566	-0.0004	+10 32 55.91	-0.012	8.8	(909) Ulla	29,31,33,34,35
097228	7 43 49.812	-0.0002	+10 31 31.04	-0.015	8.4	(909) Ulla	29,31,33,34,35
097235	7 44 14.881	0.0001	+10 30 40.77	-0.010	8.6	(909) Ulla	29,31,33,34,35
097250	7 44 49.546	0.0001	+11 27 33.73	0.008	8.9	(909) Ulla	29(s),31(s)

097251	7h 44m 49.602s	0.0001s	+11° 39′ 9.63″	-0.006″	8.9m	(909) Ulla	29(f)
097253	7 44 56.255	-0.0006	+10 9 59.17	-0.002	8.5	(909) Ulla	29,31,33
097259	7 45 14.295	-0.0008	+11 14 3.37	-0.004	8.7	(909) Ulla	29,31(f),33(s)
098313	9 0 8.357	-0.0038	+18 8 39.54	-0.025	9.3	(334) Chicago	51(f),53
098315	9 0 20.420	-0.0057	+17 46 21.77	-0.045	8.6	(334) Chicago	51(f),53
098324	9 0 55.625	-0.0006	+16 52 33.71	-0.011	9.4	(334) Chicago	51(f),53
098325	9 0 55.941	-0.0019	+18 6 11.92	-0.009	9.2	(334) Chicago	51(f),53
098328	9 1 2.710	-0.0019	+18 11 41.92	-0.028	8.8	(334) Chicago	51(f),53
098330	9 1 4.890	-0.0006	+18 2 58.37	-0.016	9.5	(334) Chicago	51(f),53
098335	9 1 33.017	-0.0005	+18 28 42.29	0.002	9.1	(334) Chicago	51,53
098346	9 2 19.721	0.0008	+16 15 7.55	-0.044	9.2	(334) Chicago	53
098358	9 2 57.300	-0.0018	+17 35 27.11	-0.007	8.0	(334) Chicago	47,49,51(c),53(c)
098362	9 3 27.558	0.0004	+17 18 48.64	-0.068	7.6	(334) Chicago	47,49,51,53
098368	9 4 8.224	-0.0024	+17 50 1.69	-0.011	9.1	(334) Chicago	47,49,51,53
098369	9 4 13.667	-0.0010	+18 28 16.61	-0.011	9.4	(334) Chicago	47,49,51,53
098371	9 4 22.701	-0.0036	+18 20 43.88	-0.044	8.4	(334) Chicago	47,49,51,53
098373	9 4 42.928	-0.0014	+16 48 44.29	-0.019	9.0	(334) Chicago	51,53

	h m s	s	° ' "	"	m		
098376	9 5 0.252	-0.0036	+17 13 46.04	-0.029	9.2	(334) Chicago	47,49,51,53
098384	9 5 46.682	-0.0010	+16 54 16.57	-0.023	7.8	(334) Chicago	47,49,51,53
098386	9 6 2.132	-0.0039	+16 37 17.90	-0.081	9.2	(334) Chicago	51,53
098387	9 6 5.460	0.0014	+17 52 52.49	-0.087	8.8	(334) Chicago	47,49,51,53
098388	9 6 6.553	-0.0038	+17 26 17.75	-0.026	8.6	(334) Chicago	47,49,51,53
098389	9 6 14.228	-0.0031	+17 40 24.18	-0.036	7.4	(334) Chicago	47(c),49(c),51,53
098399	9 7 3.000	-0.0006	+17 39 27.24	0.008	9.3	(334) Chicago	47,49
098402	9 7 6.473	-0.0015	+17 12 56.84	-0.002	9.6	(334) Chicago	47,49
098407	9 7 17.835	-0.0006	+17 1 7.75	-0.000	9.0	(334) Chicago	47,49
098413	9 7 43.965	0.0011	+17 50 30.17	-0.041	9.4	(334) Chicago	47,49
098414	9 7 56.323	-0.0021	+17 14 31.60	-0.030	8.8	(334) Chicago	47,49
098418	9 8 16.254	0.0023	+17 19 6.73	-0.070	9.0	(334) Chicago	47,49
098426	9 9 11.367	0.0004	+17 2 44.42	-0.034	9.1	(334) Chicago	47
098472	9 14 35.238	0.0007	+16 45 41.99	-0.043	8.8	(334) Chicago	43
098474	9 14 51.176	-0.0004	+16 54 55.33	-0.007	8.7	(334) Chicago	43
098478	9 15 17.275	0.0020	+16 32 1.37	-0.025	9.1	(334) Chicago	43
098483	9 15 49.310	-0.0006	+17 26 58.82	-0.016	8.6	(334) Chicago	43

098490	9 16 14.981	-0.0006	+17 22 40.48	0.022	9.2	(334) Chicago	43
098503	9 17 6.735	0.0053	+16 20 7.69	-0.049	9.3	(334) Chicago	43
098504	9 17 15.418	-0.0002	+16 8 58.69	-0.014	8.6	(334) Chicago	43
098511	9 17 51.155	-0.0009	+17 20 24.52	-0.008	8.1	(334) Chicago	43
098512	9 17 53.323	-0.0005	+16 21 52.05	-0.021	8.8	(334) Chicago	43
098513	9 17 57.723	-0.0045	+17 45 57.39	-0.090	9.0	(334) Chicago	43
098521	9 18 38.919	-0.0020	+16 48 44.17	-0.041	6.8	(334) Chicago	43(c)
098524	9 18 59.349	-0.0002	+16 40 54.38	-0.015	9.3	(334) Chicago	43
098529	9 19 38.867	-0.0010	+17 31 41.71	-0.022	9.4	(334) Chicago	43
098533	9 20 0.267	0.0013	+15 59 30.28	-0.068	8.0	(334) Chicago	43
098539	9 20 35.653	-0.0002	+16 22 48.07	-0.017	9.5	(334) Chicago	43
098561	9 22 46.441	-0.0060	+16 48 8.87	-0.020	6.3	(334) Chicago	43
099647	11 29 9.539	-0.0214	+14 38 49.64	-0.172	8.2	(909) Ulla	59,60(f)
099648	11 29 10.091	-0.0228	+14 38 35.50	-0.194	6.1	(909) Ulla	59,60(f)
099654	11 30 0.481	-0.0007	+14 0 4.56	-0.001	8.8	(909) Ulla	59(f),60(f)
099665	11 31 11.165	-0.0034	+14 45 18.71	-0.051	9.2	(909) Ulla	59,60
099666	11 31 21.145	-0.0003	+13 49 43.41	-0.033	8.1	(909) Ulla	59(f),60(f)

099667	11h 31m 32.226s	0.0035s	+15° 10′ 7.04″	-0.042″	8.8m	(909) Ulla	59,60,61(f),62(f)
099669	11 31 41.719	0.0011	+14 58 20.80	-0.036	8.7	(909) Ulla	59,60
099672	11 32 2.812	-0.0007	+17 3 47.96	-0.002	9.1	(909) Ulla	61(f),62(f)
099673	11 32 6.403	-0.0008	+17 4 24.58	-0.002	6.0	(909) Ulla	61(f),62(f)
099675	11 32 26.599	0.0003	+16 17 13.35	-0.005	9.3	(909) Ulla	59(f),60(f),61,62
099680	11 32 42.933	0.0002	+16 11 9.85	-0.066	9.3	(909) Ulla	59(f),60(f),61,62
099685	11 33 29.664	-0.0056	+16 39 31.96	0.034	9.3	(909) Ulla	61,62
099688	11 33 38.884	-0.0025	+14 58 27.28	-0.015	8.2	(909) Ulla	59(c),60(c),61,62
099694	11 34 16.720	0.0001	+14 59 43.23	-0.010	9.2	(909) Ulla	59,60,61,62
099696	11 34 21.754	0.0006	+14 17 26.05	-0.042	8.6	(909) Ulla	59,60
099701	11 35 9.733	-0.0059	+15 25 10.34	0.031	8.8	(909) Ulla	59,60,61,62
099702	11 35 22.145	-0.0013	+16 3 13.77	-0.010	8.7	(909) Ulla	59,60,61(c),62(c)
099706	11 35 34.639	0.0000	+14 49 15.21	-0.017	8.9	(909) Ulla	59,60,61(f),62(f)
099707	11 35 54.092	-0.0003	+14 12 16.11	-0.045	8.7	(909) Ulla	59,60
099708	11 35 54.663	-0.0007	+14 8 31.26	-0.002	8.9	(909) Ulla	59,60
099709	11 36 9.117	-0.0015	+14 55 43.88	0.007	9.0	(909) Ulla	59,60,61,62
099710	11 36 10.137	-0.0029	+15 41 46.77	-0.002	8.4	(909) Ulla	59,60,61,62
099712	11 36 31.648	-0.0015	+15 49 43.18	-0.035	9.0	(909) Ulla	59,60,61,62

	h m s	s	° ′ ″	″	m		
099715	11 36 51.341	0.0007	+16 48 53.39	-0.015	8.3	(909) Ulla	61,62
099716	11 36 55.290	0.0002	+14 34 30.00	-0.049	9.1	(909) Ulla	59,60
099726	11 37 55.336	-0.0000	+17 4 13.12	0.007	9.1	(909) Ulla	61(f),62(f)
099731	11 38 13.763	-0.0017	+16 32 56.74	-0.025	9.1	(909) Ulla	61,62
099734	11 38 37.897	0.0016	+14 3 48.61	-0.068	8.6	(909) Ulla	56,57(f),58
099737	11 39 3.366	0.0005	+16 57 47.99	-0.027	9.2	(909) Ulla	61(f),62(f)
099738	11 39 8.237	-0.0021	+16 57 4.23	0.022	9.0	(909) Ulla	61(f),62(f)
099740	11 39 18.280	-0.0008	+16 4 39.81	-0.032	8.9	(909) Ulla	61(f),62(f)
099745	11 39 53.080	-0.0001	+13 54 52.39	-0.048	9.1	(909) Ulla	56,57(f),58
099751	11 40 39.271	-0.0016	+13 53 8.53	-0.002	8.7	(909) Ulla	56,57(f),58
099757	11 41 6.815	-0.0025	+13 47 39.31	-0.013	9.0	(909) Ulla	56,57(f),58
099758	11 41 8.675	-0.0005	+13 47 55.92	-0.008	9.1	(909) Ulla	56,57,58
099764	11 41 44.816	0.0026	+15 14 25.06	-0.071	9.4	(909) Ulla	56,57,58
099773	11 42 31.710	-0.0048	+14 16 53.28	-0.045	8.9	(909) Ulla	56,57,58
099774	11 42 36.639	-0.0042	+14 32 27.38	-0.005	6.8	(909) Ulla	56(c),57(c),58(c)
099775	11 42 38.062	0.0001	+14 20 54.03	-0.070	9.1	(909) Ulla	56,57,58
099776	11 42 46.226	-0.0021	+13 35 22.42	-0.003	8.5	(909) Ulla	56,57,58
099779	11 43 15.525	0.0000	+13 53 48.13	-0.025	8.3	(909) Ulla	56,57,58

099780	$11^{h}\,43^{m}\,19^{s}.861$	$-0^{s}.0020$	$+15^{\circ}\,17'\,57''.66$	$-0''.034$	$8^{m}.7$	(909) Ulla	56,57,58
099784	11 43 48.703	-0.0010	+14 24 5.69	-0.024	8.5	(909) Ulla	56,57,58
099785	11 43 56.628	0.0020	+13 58 1.54	-0.046	8.4	(909) Ulla	56,57,58
099786	11 43 59.126	-0.0045	+15 14 32.32	-0.088	9.0	(909) Ulla	56,57,58
099787	11 44 15.715	-0.0002	+15 16 46.87	0.009	7.9	(909) Ulla	56,57,58
099794	11 44 51.940	-0.0032	+13 39 50.57	-0.017	8.9	(909) Ulla	56,57,58
099797	11 45 17.140	0.0009	+13 44 2.81	0.005	8.7	(909) Ulla	56,57,58
099800	11 46 4.365	-0.0074	+14 33 43.52	0.002	5.9	(909) Ulla	56,57,58
099801	11 46 9.310	-0.0025	+13 33 4.75	-0.045	9.5	(909) Ulla	56,57,58
099827	11 48 21.444	-0.0086	+12 33 23.23	0.009	6.2	(909) Ulla	52,54
099829	11 48 51.024	-0.0175	+12 4 59.64	0.002	7.0	(909) Ulla	52,54
099833	11 49 43.258	-0.0029	+11 15 56.12	0.001	8.4	(909) Ulla	52,54
099846	11 51 3.025	0.0012	+11 15 56.86	-0.033	9.0	(909) Ulla	52,54
099849	11 51 19.427	0.0000	+11 23 51.74	0.009	8.3	(909) Ulla	52,54
099852	11 51 32.626	0.0036	+12 20 10.74	-0.116	8.2	(909) Ulla	52(c),54(c)
099862	11 52 21.912	-0.0099	+11 58 21.76	0.020	8.6	(909) Ulla	52,54
099868	11 53 4.629	-0.0118	+12 15 51.25	-0.024	9.2	(909) Ulla	52,54

099872	11 53 14.856	-0.0013	+11 31 53.18	0.020	8.8	(909) Ulla	52,54
099877	11 53 59.191	-0.0009	+12 37 37.22	-0.033	8.7	(909) Ulla	52,54
099881	11 54 36.919	-0.0060	+12 1 58.32	-0.100	9.1	(909) Ulla	52,54
099882	11 54 39.523	-0.0036	+13 13 34.88	-0.011	9.1	(909) Ulla	52,54
099889	11 56 6.111	-0.0015	+13 7 59.78	-0.021	9.2	(909) Ulla	50
099892	11 56 29.184	-0.0010	+12 20 59.41	-0.023	8.7	(909) Ulla	50
099895	11 56 48.961	0.0007	+13 10 26.49	-0.016	8.9	(909) Ulla	50
099897	11 56 54.747	-0.0013	+12 7 44.99	-0.008	8.8	(909) Ulla	50
099905	11 58 15.100	-0.0027	+12 18 20.04	-0.019	8.7	(909) Ulla	50
099906	11 58 17.082	-0.0012	+12 15 42.57	-0.012	8.8	(909) Ulla	50
099908	11 58 29.756	-0.0006	+13 17 13.13	-0.033	8.7	(909) Ulla	50
099910	11 58 40.115	-0.0027	+12 39 21.98	-0.005	6.9	(909) Ulla	50(c)
099918	11 59 56.296	-0.0006	+11 47 7.89	-0.022	9.1	(909) Ulla	50
099920	12 0 5.380	0.0001	+12 9 21.37	-0.011	8.7	(909) Ulla	50
099924	12 0 19.961	-0.0051	+13 5 9.70	-0.018	7.8	(909) Ulla	50
099925	12 0 44.013	-0.0024	+13 17 51.25	-0.075	9.0	(909) Ulla	50
099929	12 2 17.958	0.0003	+11 55 17.13	-0.018	9.2	(909) Ulla	50

099939	$12^h 3^m 32.^s617$	$0.^s0001$	$+10^\circ 45' 37.''75$	$-0.''013$	$8.^m5$	(909) Ulla	46
099946	12 4 42.741	-0.0010	+10 48 38.36	-0.078	8.4	(909) Ulla	46
113597	6 10 30.779	-0.0002	+ 9 36 44.60	0.018	8.5	(107) Camilla	114
113608	6 11 6.968	0.0019	+ 9 17 7.21	-0.002	8.6	(107) Camilla	111
113620	6 11 51.949	0.0017	+ 9 45 36.98	-0.023	8.3	(107) Camilla	114
113643	6 12 38.850	-0.0012	+ 9 46 34.35	-0.013	8.6	(107) Camilla	111
113646	6 12 52.035	-0.0009	+ 9 2 7.74	-0.070	8.8	(107) Camilla	111
113662	6 13 32.237	0.0001	+ 8 29 6.89	0.001	8.8	(107) Camilla	111(f)
113675	6 14 21.142	0.0000	+ 9 57 44.32	-0.063	5.3	(107) Camilla	114
113685	6 14 51.339	-0.0006	+ 8 54 17.20	-0.013	8.6	(107) Camilla	111
113694	6 15 11.952	-0.0008	+ 9 32 11.86	-0.004	9.2	(107) Camilla	111,114
113699	6 15 20.862	-0.0016	+ 9 20 28.57	-0.021	8.4	(107) Camilla	111(c)
113705	6 15 39.029	-0.0006	+ 8 33 1.44	0.008	8.8	(107) Camilla	111
113724	6 16 46.232	-0.0018	+ 9 2 24.11	0.001	8.6	(107) Camilla	111
113726	6 16 56.733	-0.0012	+ 9 9 44.45	0.002	8.8	(107) Camilla	111(f)
113746	6 18 18.751	-0.0006	+ 9 47 9.19	0.009	8.3	(107) Camilla	111
113756	6 18 44.496	-0.0018	+ 8 43 50.66	-0.008	8.2	(107) Camilla	111

113773	$6^h 19^m 24^s.393$	$0^s.0002$	$+ 9° 11' 1''.11$	$-0''.015$	$8^m.4$	(107)	Camilla	111
113781	6 19 43.096	-0.0001	+ 8 56 45.87	-0.059	8.9	(107)	Camilla	111(f)
113916	6 24 58.064	-0.0016	+ 9 39 30.15	-0.012	8.2	(107)	Camilla	107
113932	6 26 4.612	-0.0026	+ 9 35 35.31	-0.012	8.6	(107)	Camilla	107
113957	6 27 21.272	-0.0013	+ 9 3 51.13	-0.027	6.5	(107)	Camilla	107
113970	6 27 44.028	-0.0013	+ 9 16 48.75	0.010	8.5	(107)	Camilla	107
113986	6 28 29.670	-0.0011	+ 9 49 35.76	-0.007	8.6	(107)	Camilla	100,104,107
113994	6 28 46.583	-0.0016	+ 9 15 13.33	0.010	8.3	(107)	Camilla	104(s),107
114027	6 29 51.151	-0.0005$	+ 9 31 44.88	-0.000	8.6	(107)	Camilla	100,104,107
114049	6 31 1.443	0.0004	+ 9 12 30.22	-0.021	8.5	(107)	Camilla	100,104
114070	6 31 39.781	-0.0007	+ 9 7 31.70	-0.011	8.3	(107)	Camilla	100(f),104
114081	6 31 59.780	-0.0007	+ 9 48 8.86	-0.004	8.7	(107)	Camilla	104(s)
114082	6 32 0.149	-0.0006	+ 9 25 36.10	-0.011	8.8	(107)	Camilla	100(f),104
114095	6 32 38.649	-0.0017	+ 9 53 18.98	-0.008	8.6	(107)	Camilla	100(f),104(f)
114144	6 34 38.920	-0.0004	+ 9 24 14.94	-0.009	8.6	(107)	Camilla	100,104
114147	6 34 47.603	-0.0046	+ 9 11 20.93	-0.088	8.3	(107)	Camilla	104(f)
114149	6 34 53.386	-0.0006	+ 8 57 48.65	-0.008	8.6	(107)	Camilla	100

	h m s	s	° ′ ″	″	m		
114202	6 36 24.824	-0.0021	+ 9 34 31.69	-0.007	8.3	(107) Camilla	100,104(f)
114204	6 36 26.225	-0.0008	+ 9 41 29.92	-0.019	8.2	(107) Camilla	104(s)
114220	6 36 50.461	-0.0014	+ 9 47 55.64	-0.023	8.2	(107) Camilla	104(f)
114439	6 45 45.331	0.0015	+ 8 40 43.57	0.017	8.7	(107) Camilla	98
114444	6 45 52.492	-0.0002	+ 9 25 28.96	-0.017	8.8	(107) Camilla	98
114451	6 46 4.606	0.0015	+ 8 52 0.76	-0.026	8.9	(107) Camilla	98
114468	6 46 36.889	-0.0011	+ 9 38 22.82	-0.000	8.4	(107) Camilla	98
114474	6 47 0.337	-0.0007	+ 8 27 57.80	-0.022	8.5	(107) Camilla	98
114497	6 47 54.145	-0.0015	+ 9 19 5.90	0.002	8.6	(107) Camilla	98
114504	6 48 31.495	0.0010	+ 9 6 18.48	-0.068	8.5	(107) Camilla	98
114509	6 48 33.889	0.0002	+ 8 34 17.35	0.018	8.6	(107) Camilla	98
114515	6 48 51.145	0.0003	+ 9 30 3.19	-0.027	7.6	(107) Camilla	98(c)
114552	6 49 58.677	-0.0005	+ 8 45 23.54	0.016	8.6	(107) Camilla	98
114562	6 50 27.633	-0.0005	+ 9 51 26.47	-0.006	8.7	(107) Camilla	98
114590	6 51 46.932	-0.0006	+ 9 33 1.94	0.002	8.5	(107) Camilla	98
114628	6 52 58.400	-0.0007	+ 9 3 34.58	-0.007	8.6	(107) Camilla	98
114635	6 53 5.013	-0.0027	+ 9 26 15.09	0.012	8.7	(107) Camilla	98

115891	$7^h 42^m 54^s.299$	$0^s.0011$	$+9^\circ 58' 53''.98$	$-0''.043$	$8^m.5$	(909) Ulla	29,31,33,34,35
115932	7 44 48.780	-0.0027	+ 9 45 16.63	-0.133	7.2	(909) Ulla	29,31
115950	7 45 27.521	-0.0004	+ 9 50 32.22	-0.029	8.9	(909) Ulla	29(s),31(f),33(s)
116080	7 51 1.722	0.0017	+ 9 7 58.72	-0.054	8.4	(909) Ulla	24(f),25(f)
116103	7 52 12.847	-0.0023	+ 8 0 57.92	-0.001	9.0	(909) Ulla	24,25,27
116112	7 52 27.606	-0.0021	+ 9 28 57.34	-0.010	7.1	(909) Ulla	24(f),25(f)
116114	7 52 34.102	-0.0052	+ 8 34 36.97	0.014	9.2	(909) Ulla	24,25,27
116119	7 52 47.880	-0.0008	+ 8 51 16.82	-0.020	8.6	(909) Ulla	24,25,27
116120	7 52 48.470	-0.0010	+ 8 59 49.65	-0.089	5.8	(909) Ulla	24(f),25(f),27
116126	7 53 3.825	-0.0033	+ 9 18 31.81	0.035	8.9	(909) Ulla	24(f),25(f)
116131	7 53 15.845	-0.0004	+ 9 33 14.99	-0.004	8.8	(909) Ulla	24(f),25(f)
116133	7 53 17.257	-0.0018	+ 8 17 14.57	-0.029	8.6	(909) Ulla	24,25,27
116141	7 53 42.133	-0.0019	+ 9 20 38.89	-0.012	9.1	(909) Ulla	24(f),25(f)
116154	7 54 3.863	-0.0019	+ 9 10 57.27	-0.030	8.5	(909) Ulla	24(f),25(f),27
116157	7 54 7.045	-0.0010	+ 8 42 14.40	0.001	8.7	(909) Ulla	24,25,27
116158	7 54 7.820	0.0007	+ 7 50 34.06	-0.042	8.9	(909) Ulla	24,25,27
116162	7 54 33.237	0.0003	+ 8 46 35.31	-0.019	6.1	(909) Ulla	24(c),25(c),27(c)

116163	7 54 34.235	-0.0012	+ 8 32 23.08	0.015	8.7	(909) Ulla	24,25,27
116164	7 54 34.839	-0.0013	+ 8 6 25.63	-0.009	8.7	(909) Ulla	24,25,27
116167	7 54 43.221	0.0002	+ 9 37 8.93	-0.042	8.9	(909) Ulla	24(f),25(f)
116172	7 54 57.350	-0.0006	+ 9 11 4.64	-0.006	9.0	(909) Ulla	24,25,27
116177	7 55 14.974	-0.0030	+ 7 58 12.06	0.011	9.0	(909) Ulla	24,25,27
116193	7 56 10.356	0.0031	+ 8 35 23.05	0.020	8.5	(909) Ulla	24,25,27
116196	7 56 13.682	0.0016	+ 8 39 25.69	0.016	8.8	(909) Ulla	24,25,27
116198	7 56 14.453	-0.0008	+ 8 30 28.20	0.042	9.5	(909) Ulla	24,25,27
116199	7 56 20.026	0.0006	+ 8 17 9.31	-0.006	8.5	(909) Ulla	24,25,27
116200	7 56 28.264	0.0000	+ 8 5 46.16	-0.005	8.4	(909) Ulla	24,25,27
116203	7 56 32.273	-0.0033	+ 9 1 25.01	0.003	9.3	(909) Ulla	24,25,27
116219	7 57 35.912	-0.0019	+ 9 6 43.51	0.025	8.2	(909) Ulla	24,25
116220	7 57 37.020	-0.0012	+ 8 13 22.23	-0.010	8.6	(909) Ulla	24,25,27
116241	7 58 31.257	-0.0025	+ 8 30 56.89	-0.051	8.4	(909) Ulla	24,25,27
116503	8 11 11.948	0.0009	+ 6 27 19.51	-0.026	8.6	(909) Ulla	12,13,14
116506	8 11 17.888	-0.0010	+ 7 0 13.12	-0.011	8.5	(909) Ulla	14(f)
116508	8 11 18.967	-0.0015	+ 6 50 58.08	0.022	8.6	(909) Ulla	14

116511	8ʰ 11ᵐ 48.159ˢ	-0.0017ˢ	+ 5° 46′ 15.43″	-0.025″	8.7ᵐ	(909) Ulla	12,13,14
116517	8 11 58.896	-0.0019	+ 5 27 14.94	-0.006	8.5	(909) Ulla	14(f)
116522	8 12 2.608	-0.0021	+ 7 7 13.59	0.012	9.0	(909) Ulla	12(s),13,14,15,16
116523	8 12 2.776	0.0015	+ 5 34 18.56	-0.027	8.6	(909) Ulla	14
116527	8 12 10.071	-0.0012	+ 6 58 13.22	-0.031	8.3	(909) Ulla	12,13,14,15,16
116533	8 12 26.524	-0.0014	+ 7 13 8.83	-0.009	8.6	(909) Ulla	12(f),14
116534	8 12 27.600	-0.0001	+ 5 34 58.16	-0.010	8.7	(909) Ulla	12,13,14,15,16
116564	8 13 38.767	-0.0024	+ 6 16 6.74	-0.007	8.5	(909) Ulla	12,13,14,15,16
116565	8 13 40.826	-0.0010	+ 6 26 32.21	-0.021	8.5	(909) Ulla	12,13,14,15,16
116567	8 13 45.568	-0.0015	+ 7 12 46.60	0.014	9.2	(909) Ulla	12,13,14
116572	8 14 0.377	-0.0032	+ 5 18 16.04	0.004	8.9	(909) Ulla	12,13
116577	8 14 21.685	-0.0012	+ 7 13 31.92	-0.007	8.8	(909) Ulla	12,13,14
116587	8 14 58.034	-0.0004	+ 6 24 3.74	-0.017	8.5	(909) Ulla	12,14,15,16
116589	8 15 1.672	-0.0018	+ 6 10 41.98	0.014	9.1	(909) Ulla	12,13,14,15,16
116596	8 15 16.525	-0.0013	+ 6 37 29.17	-0.003	9.1	(909) Ulla	12,15,16
116597	8 15 19.423	-0.0019	+ 6 23 23.96	0.001	7.0	(909) Ulla	12(c),13(c),14(c), 15(c),16(c)
116604	8 15 43.345	-0.0007	+ 6 7 42.51	0.022	8.7	(909) Ulla	12(s),13,14,15,16
116612	8 16 5.510	-0.0012	+ 7 12 16.59	0.020	8.6	(909) Ulla	12,13,14(f),15,16

	h m s	s	° ' "	"	m		
116614	8 16 24.451	-0.0013	+ 5 31 26.34	0.006	9.0	(909) Ulla	12,13,14
116616	8 16 30.196	0.0005	+ 5 51 37.46	-0.025	8.6	(909) Ulla	12,13,14,15,16
116617	8 16 31.959	-0.0046	+ 6 56 56.39	0.034	8.5	(909) Ulla	12,13,14,15,16
116623	8 16 54.844	-0.0007	+ 5 48 35.69	-0.003	8.6	(909) Ulla	12,13,14,15,16
116643	8 17 43.178	-0.0007	+ 5 49 10.51	-0.025	8.8	(909) Ulla	12,13,14,15,16
116650	8 18 10.030	-0.0009	+ 6 59 34.01	-0.024	8.4	(909) Ulla	14(f),15,16
116653	8 18 16.722	-0.0007	+ 6 39 46.51	-0.011	8.6	(909) Ulla	12,13,14(f),15,16
116654	8 18 18.002	-0.0003	+ 6 37 43.89	-0.002	8.7	(909) Ulla	14(f),15,16
116658	8 18 23.021	-0.0027	+ 5 30 0.02	0.028	8.8	(909) Ulla	14(f)
116662	8 18 32.072	-0.0050	+ 7 9 4.48	0.144	9.0	(909) Ulla	14(f)
119180	12 0 1.370	-0.0008	+ 8 20 54.25	-0.041	8.4	(909) Ulla	42,44
119182	12 0 11.727	-0.0051	+ 8 6 7.41	-0.018	8.6	(909) Ulla	42,44
119184	12 0 24.247	0.0039	+ 8 59 35.32	-0.086	8.6	(909) Ulla	42,44
119187	12 0 50.843	0.0020	+ 9 31 50.52	-0.036	8.5	(909) Ulla	42,44
119192	12 1 33.228	0.0008	+ 9 58 38.00	-0.023	8.6	(909) Ulla	42,44,46
119198	12 1 44.717	-0.0164	+ 9 28 15.12	0.030	9.1	(909) Ulla	42,44,46
119208	12 2 19.877	0.0003	+ 9 27 1.19	-0.006	9.2	(909) Ulla	42,44,46

	h m s	s	° ′ ″	″	m		
119209	12 2 21.683	-0.0039	+ 8 59 36.40	0.040	8.7	(909) Ulla	42(c),44(c)
119210	12 2 30.913	0.0036	+ 9 31 6.34	-0.055	8.2	(909) Ulla	42,44,46
119213	12 2 39.704	-0.0149	+ 9 0 38.34	0.043	4.2	(909) Ulla	42,44,46
119224	12 4 37.509	-0.0046	+ 9 56 28.64	-0.008	7.3	(909) Ulla	42,44,46(c)
119246	12 7 8.814	-0.0055	+ 9 34 9.86	0.020	9.0	(909) Ulla	46
119248	12 7 20.538	-0.0010	+ 9 41 14.98	-0.052	9.1	(909) Ulla	46

Notes:

(c) This star was the centre star for this plate.

(f) This star was measured only on the first measurement of this plate.

(s) This star was measured only on the second measurement of this plate.

(*) This star was the centre star for all these plates, but its measurements were too inaccurate to be used in the reductions.

Right ascension α and declination δ are given with respect to the equator and equinox of 1950.0.

The columns μ_α and μ_δ give the proper motion in α and δ respectively, in seconds of time and of arc respectively, per annum.

Magnitudes "m" are visual magnitudes m_v, as photographic magnitudes m_{pg} are not given in the SAO Star Atlas for all the stars.

Table 3: Measured positions of minor planets

plate	date	U.T.	R.A.	$d\alpha$	dec.	$d\delta$	total error
			(909) Ulla 1971-1972				
12	22 Dec 1971	$01^h 20^m 09^s$	$8^h 14^m 10^s.721$	$0^s.034$	$+6^\circ 13' 44''.13$	$0''.51$	$0''.72$
13	22 Dec 1971	02 03 02	8 14 9.880	0.014	+ 6 13 47.85	0.21	0.30
14	22 Dec 1971	04 47 42	8 14 6.520	0.018	+ 6 14 3.32	0.22	0.35
15	23 Dec 1971	02 23 31	8 13 40.192	0.027	+ 6 16 13.53	0.80	0.89
16	24 Dec 1971	04 53 3	8 13 6.701	0.036	+ 6 19 2.43	0.38	0.66
24	20 Jan 1972	02 38 56	7 55 31.486	0.034	+ 8 12 27.72	0.15	0.53
25	20 Jan 1972	03 24 49	7 55 30.236	0.032	+ 8 12 39.00	0.20	0.52
27	20 Jan 1972	23 52 47	7 54 53.195	0.046	+ 8 17 25.44	0.51	0.86
29	10 Feb 1972	23 43 40	7 41 4.263	0.022	+10 26 18.66	0.12	0.35
31	11 Feb 1972	22 00 01	7 40 34.060	0.025	+10 32 14.65	0.28	0.47
33	12 Feb 1972	21 8 13	7 40 3.416	0.015	+10 38 23.05	0.15	0.27
33	12 Feb 1972	21 41 8	7 40 2.754	0.019	+10 38 32.19	0.30	0.41
34	13 Feb 1972	20 40 21	7 39 33.201	0.029	+10 44 38.34	0.27	0.51

35	14 Feb 1972	20ʰ 6ᵐ 30ˢ	7ʰ 39ᵐ 3ˢ.853	0ˢ.025	+10° 50′ 51″.86	0″.30	0″.48

(909) Ulla 1972-1973

42	7 Feb 1973	00 37 19	12 5 5.637	0.196	+ 9 3 35.77	3.37	4.47
44	7 Feb 1973	01 54 06	12 5 4.581	0.230	+ 9 3 59.54	5.52	6.51
46	10 Feb 1973	00 42 28	12 4 15.173	0.086	+ 9 23 57.25	0.58	1.41
50	28 Feb 1973	04 14 56	11 56 28.589	0.073	+11 38 42.41	1.01	1.49
52	4 Mar 1973	22 57 5	11 53 50.746	0.145	+12 17 30.81	0.88	2.35
54	8 Mar 1973	3 18 32	11 51 56.584	0.080	+12 39 27.61	0.35	1.25
56	24 Mar 1973	22 11 21	11 41 39.532	0.014	+14 35 31.68	0.11	0.24
57	24 Mar 1973	22 31 18	11 41 38.981	0.013	+14 35 36.75	0.13	0.23
58	25 Mar 1973	22 34 45	11 41 2.956	0.021	+14 41 39.91	0.19	0.37
59	31 Mar 1973	02 22 01	11 38 2.047	0.148	+15 10 57.00	0.94	2.41
60	31 Mar 1973	02 42 57	11 38 1.551	0.190	+15 11 3.67	0.68	2.93
61	5 Apr 1973	02 23 18	11 35 18.389	0.020	+15 35 55.82	0.16	0.34
62	5 Apr 1973	02 44 14	11 35 17.910	0.024	+15 35 59.20	0.14	0.39

(334) Chicago 1971-1972

17	12 Jan 1972	00 52 40	5 40 35.546	0.954	+19 32 21.66	4.97	15.15
19	14 Jan 1972	21 41 28	5 38 57.702	0.220	+19 34 37.48	0.61	3.36
20	14 Jan 1972	22 25 21	5 38 56.938	0.347	+19 34 38.66	0.89	5.28
22	19 Jan 1972	23 43 25	5 36 20.955	0.129	+19 38 50.01	0.84	2.11
23	20 Jan 1972	00 17 19	5 36 20.245	0.026	+19 38 52.34	0.78	0.87
23	20 Jan 1972	00 54 43	5 36 19.707	0.025	+19 38 54.33	0.63	0.73
26	20 Jan 1972	23 07 54	5 35 53.624	0.024	+19 39 43.04	0.37	0.52

(334) Chicago 1972-1973

43	7 Feb 1973	01 20 12	9 17 34.262	0.022	+16 27 5.78	0.10	0.34
47	27 Feb 1973	03 49 56	9 4 59.907	0.017	+17 37 36.45	0.50	0.56
49	28 Feb 1973	03 40 01	9 4 27.367	0.018	+17 40 35.29	0.26	0.37
51	4 Mar 1973	22 17 12	9 2 0.876	0.033	+17 54 2.39	0.48	0.69
53	8 Mar 1973	02 47 37	9 0 32.701	0.041	+18 2 9.73	0.41	0.74

(1574) Meyer 1972-1973

36	8 Sep 1972	00 16 8	0 37 33.739	0.109	+21 47 13.90	1.53	2.24

38	8 Sep 1972	01h 29m 57s	0h 37m 32s.040	0s.116	+21° 47′ 7″.09	0″.71	1″.88
40	9 Sep 1972	23 22 25	0 36 31.729	0.085	+21 43 10.80	0.31	1.31

(414) Liriope 1973-1974

81	22 Oct 1973	21 53 05	4 57 22.472	0.085	+12 58 57.59	0.79	1.50
82	22 Oct 1973	22 23 30	4 57 22.225	0.061	+12 58 56.70	0.52	1.05
83	26 Oct 1973	23 00 37	4 56 12.178	0.048	+12 54 1.65	0.37	0.81
85	27 Oct 1973	00 02 57	4 56 11.274	0.017	+12 53 59.74	0.38	0.46
86	29 Oct 1973	02 06 44	4 55 26.465	0.042	+12 51 31.56	0.25	0.68
88	23 Nov 1973	04 34 16	4 40 18.809	0.230	+12 30 59.73	1.39	3.72
97	28 Nov 1973	00 25 44	4 36 31.690	0.030	+12 30 12.30	0.93	1.03
101	24 Dec 1973	00 12 14	4 17 4.768	0.033	+12 52 56.30	0.19	0.53
110	15 Jan 1974	19 45 38	4 8 7.956	0.058	+13 53 40.68	0.64	1.08

(87) Sylvia 1973-1974

90	23 Nov 1973	05 32 06	6 40 0.219	0.088	+26 59 30.12	0.38	1.37
99	28 Nov 1973	01 21 34	6 37 29.582	0.065	+27 15 34.00	0.26	1.01
102	24 Dec 1973	00 45 40	6 17 47.956	0.035	+28 34 58.34	0.15	0.55

105	26 Dec 1973	01 14 42	6 16 2.069	0.029	+28 39 57.48	0.36	0.56
108	30 Dec 1973	01 39 22	6 12 32.248	0.035	+28 49 7.82	0.27	0.59
112	15 Jan 1974	20 40 24	5 59 10.105	0.054	+29 16 36.14	0.27	0.85
115	17 Jan 1974	19 40 46	5 57 50.547	0.031	+29 18 42.81	0.15	0.49

(107) Camilla 1973-1974

98	28 Nov 1973	00 54 39	6 48 54.400	0.033	+ 9 37 44.04	0.57	0.75
100	23 Dec 1973	04 10 23	6 33 15.149	0.023	+ 9 18 36.33	0.20	0.40
104	26 Dec 1973	00 41 47	6 31 4.140	0.035	+ 9 20 30.84	0.07	0.53
107	30 Dec 1973	01 12 16	6 27 57.551	0.023	+ 9 24 39.39	0.15	0.38
111	15 Jan 1974	20 17 03	6 15 43.171	0.047	+ 9 58 33.51	0.65	0.96
114	17 Jan 1974	19 14 00	6 14 28.535	0.029	+10 4 3.23	0.31	0.53

Table 4a: "Reference" stars in Pleiades

no.	R.A.	dec.	m(pg)	B-V	$d\alpha$	$d\delta$
248	$3^h 38^m 5^s.639$	$23° 13' 39''.15$	$11^m.79$	$0^m.84$	$0''.02$	$-0''.26$
285	3 38 14.921	24 25 16.98	11.87	0.58	0.02	0.22
294	3 38 16.062	24 5 17.15	11.03	0.73	0.24	0.06
343	3 38 28.340	23 21 11.17	12.28	1.32	-0.05	0.30
388	3 38 39.054	23 44 50.20	13.27	1.82	-0.12	0.23
404	3 38 42.169	24 11 43.29	11.70	0.69	-0.04	-0.06
430	3 38 47.357	23 54 58.44	12.17	0.98	-0.01	-0.17
457	3 38 53.189	23 16 29.89	11.32	1.65	-0.04	0.27
476	3 38 57.898	23 36 24.05	11.61	0.85	-0.03	-0.20
546	3 39 12.574	24 10 12.31	11.51	0.77	0.12	-0.15
570	3 39 16.076	23 48 38.78	12.98	1.27	-0.42	0.06
589	3 39 19.185	24 34 28.62	12.69	0.42	-0.25	0.70
593	3 39 19.793	24 24 37.75	12.25	0.75	-0.08	0.09
649	3 39 29.496	23 10 21.81	12.84	0.83	0.15	-0.02
738	3 39 43.739	23 26 28.45	13.42	1.12	0.30	0.04

	h m s	° ′ ″	m	m	″	″
746	3 39 44.632	24 7 6.41	12.06	0.94	-0.11	-0.10
750	3 39 46.402	23 57 25.00	12.49	0.79	-0.02	-0.20
803	3 39 55.320	23 18 28.99	13.39	1.30	-0.18	0.10
812	3 39 56.191	24 29 20.48	13.02	0.91	-0.00	0.28
903	3 40 11.344	23 44 56.69	11.25	0.72	0.21	0.10
916	3 40 14.017	24 18 36.95	12.43	1.00	0.15	-0.23
975	3 40 22.836	23 10 29.33	11.41	0.84		
1021	3 40 29.378	23 0 41.16	11.68	1.26		
1032	3 40 31.071	24 7 20.68	11.95	0.82		
1070	3 40 35.941	23 33 1.44	11.62	0.74	0.22	0.02
1117	3 40 41.791	23 28 35.81	10.95	0.82		
1149	3 40 47.100	24 0 6.03	10.94	0.45	0.10	-0.38
1207	3 40 56.659	24 29 8.74	11.10	0.72	0.07	-0.02
1242	3 41 2.328	23 46 42.87	12.79	0.68	-0.17	-0.05
1275	3 41 6.145	23 11 4.46	12.34	0.96		
1337	3 41 18.350	23 52 59.09	12.99	0.67	0.48	-0.04
1346	3 41 20.197	23 30 31.52	11.65	0.90	-0.09	0.12
1391	3 41 27.297	23 14 40.58	11.53	0.61	-0.50	0.26

	h m s	° ′ ″	m	m	″	″
1474	3 41 37.799	24 13 35.87	11.90	1.66	-0.16	-0.10
1514	3 41 43.083	24 3 19.64	11.17	0.72	0.26	-0.36
1521	3 41 43.373	23 41 38.37	12.01	1.44	-0.31	0.19
1640	3 42 0.060	23 23 14.19	12.29	0.73	0.18	0.31
1645	3 42 0.261	23 9 23.23	12.08	0.83	0.11	0.20
1680	3 42 4.980	23 46 45.48	13.48	1.67	-0.27	0.14
1794	3 42 20.756	23 34 57.17	11.03	0.69	-0.12	-0.15
1826	3 42 26.483	24 34 13.49	11.34	0.61	0.19	0.13
1855	3 42 29.144	23 59 59.83	12.49	0.40	0.22	-0.25
1942	3 42 41.244	24 22 31.07	12.39	0.71	-0.14	-0.06
1949	3 42 42.689	24 8 53.67	12.40	0.42	-0.12	-0.25
1955	3 42 43.345	23 28 16.30	13.29	1.62	-0.34	0.60
1967	3 42 44.736	22 59 31.31	12.64	0.59	-0.08	-0.22
2017	3 42 50.282	23 13 6.37	11.59	1.53	0.04	0.48
2068	3 42 58.003	23 50 5.16	12.12	0.70		
2079	3 42 59.355	23 43 1.91	12.63	1.45		
2182	3 43 14.011	23 18 23.03	11.30	0.36	0.10	0.28
2229	3 43 20.320	24 6 31.67	12.34	0.62		

	h m s	° ′ ″	″	″	″	″
2244	3 43 22.024	24 28 15.44	13.52	1.13		
2272	3 43 25.929	24 33 52.02	14.27	0.04		
2329	3 43 35.193	24 19 27.36	11.25	1.39		
2341	3 43 36.799	23 29 24.54	11.61	0.79	0.28	0.05
2366	3 43 39.058	23 59 27.17	12.30	1.05		
2401	3 43 43.713	23 43 55.93	11.81	1.15		
2406	3 43 44.209	22 59 5.53	11.85	0.81	-0.12	-0.61
2462	3 43 54.207	23 24 3.34	12.39	0.98		
2485	3 43 57.544	23 13 33.06	11.30	1.42	0.16	0.18
2554	3 44 8.752	23 54 39.01	12.04	0.65		
2564	3 44 10.198	23 42 53.23	13.46	1.36		
2571	3 44 11.023	23 49 8.35	12.17	1.14		
2597	3 44 14.661	24 19 28.18	11.85	0.59		
2600	3 44 15.037	24 29 17.31	13.90	0.84		
2644	3 44 22.919	24 9 46.97	11.79	0.79		
2648	3 44 23.385	23 34 40.62	13.19	0.64		
2693	3 44 30.138	24 2 39.13	12.82	1.28		

Table 4b: "Unknown" stars in Pleiades

no.	R.A.	dec.	m(pg)	B−V	dα	dδ
	h m s	° ′ ″	m	m	″	″
468	3 38 56.168	23 47 56.11	3.52	−0.12	−0.28	−0.87
688	3 39 35.937	23 18 26.04	13.67	0.90	+0.03	−0.18
706	3 39 38.649	24 11 12.05	12.57	0.66	+0.38	−0.01
1058	3 40 33.832	24 33 32.27	10.29	0.36	+0.28	−0.02
1122	3 40 42.724	23 47 31.94	9.74	0.49	−0.16	−0.38
1349	3 41 20.831	23 36 14.62	12.36	1.65	+0.03	−0.08
1380	3 41 24.903	23 29 37.01	6.94	−0.01	+0.57	+0.20
1432	3 41 32.365	23 47 45.00	2.73	−0.11	−0.52	−0.52
1538	3 41 45.975	23 58 22.25	12.55	0.74	−0.22	−0.35
1809	3 42 22.431	23 45 12.95	13.22	1.30	−0.14	−0.32
2181	3 43 14.202	23 49 51.80	5.28	0.11	−0.57	−1.24
2311	3 43 32.624	23 24 24.23	12.17	0.96	−0.03	+0.33
2407	3 43 44.360	24 9 29.06	13.25	1.26	+0.45	−0.18

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